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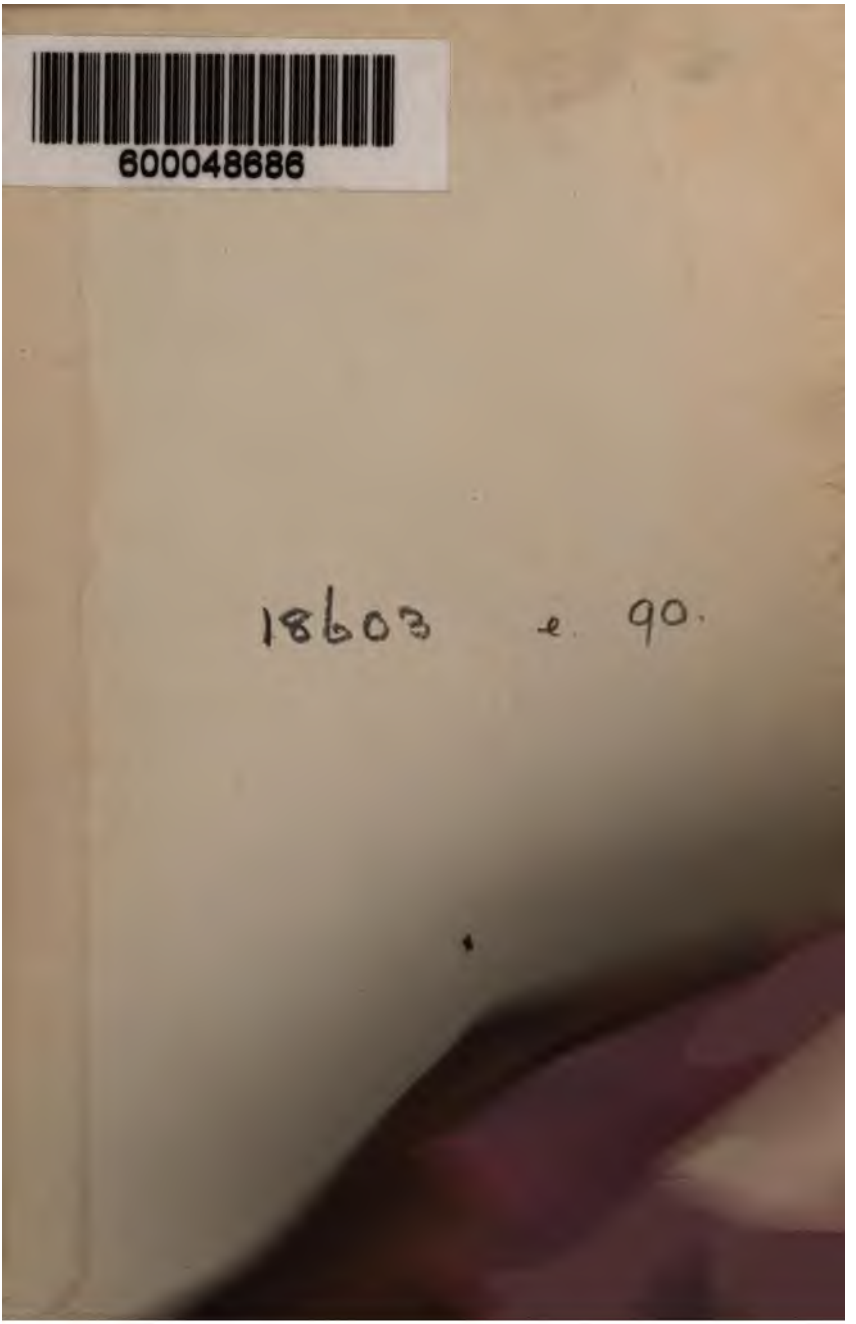
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THE

STUDENT'S MECHANICS.

WORKS

BY

W. J. MACQUORN RANKINE, C.E., LL.D., F.R.S.,

Late Regius Professor of Engineering and Mechanics in the University of Glasgow.

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THE

STUDENT'S MECHANICS:

AN INTRODUCTION TO THE STUDY

OF

FORCE AND MOTION.

BY

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With Appendix of Examples Worked and Unworked.

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PREFACE.

THIS treatise has grown out of a series of articles contributed to *The Engineer* in 1881, under the title, "The Foundations of Mechanics." It differs accordingly from the many previous works on the subject, mainly in the fulness and care with which these foundations have been considered. The successful prosecution of Mechanics, especially as applied to practical construction, chiefly depends on the obtaining a clear and thorough mastery of a few leading principles (*e.g.*, the composition of forces, the principle of moments, the doctrine of energy) which are alone necessary for the solution of almost all the problems of ordinary practice. It is, of course, easy to learn such propositions sufficiently for the purposes of an examination: it is by no means easy to know and understand them so thoroughly as to be able to use them freely and confidently in attacking questions of practical importance. It is in the hope of facilitating, in some cases, the attainment of such knowledge that these pages have been written.

With this view the *deductive* character of Mechanics has been indicated at the outset: and the ultimate definitions and axioms whereon the science rests have been examined and stated with special care and fulness. With the same object of thoroughness two principles, which have usually been reserved for a higher stage of progress, are here brought forward and placed in the

forefront, where, in the writer's opinion, they should properly stand. The doctrine of Central Forces, as representing the general scheme of nature, though assumed, explicitly or implicitly, in all treatises dealing with the higher branches of the subject, is not usually adopted from the beginning, as in this work; and the way in which it connects and explains all the fundamental laws of the science is not therefore recognised. Again, the conception of Energy is not generally introduced until the end of an elementary treatise, if at all: whereas here the fundamental fact, that the energy exerted represents the effect of a given force acting for a given time, is explained at the outset, and the consequences which flow from it are developed throughout the book.

On these matters, and on some others, such as the laws of motion, centrifugal force, elasticity, &c., it is believed that some novelty and freshness of treatment will be recognised; although the writer is far from claiming any actual originality for his methods. No such novelty is, of course, to be looked for or desired in the proofs of the ordinary propositions which make up the bulk of elementary Statics and Kinematics; and the writer's thanks are due to the Bishop of Carlisle for permission to follow, in most of these, the admirably clear demonstrations to be found in "Goodwin's Course of Mathematics." In a few cases he has adopted substantially the treatment of other writers, such as Prof. Goodeve in his well-known "Principles of Mechanics," and Mr. Philip Magnus in his "Lessons on Elementary Mechanics." The collection of examples at the end is not meant to present any features either of originality or difficulty: they are simply to be regarded as easy exercises, tending to the attainment of that grasp over Principles which it is the special aim of the book to impart. With this addition, it is hoped that

it may be found useful at once to youths commencing their study of the subject under regular instruction, and to older students who may wish to acquire by themselves a fuller knowledge of its principles, with the view of following them up either into the higher branches of the theory, or into the practical region of Applied Mechanics. A few propositions, for which some knowledge of the Calculus, or of Solid Geometry, is required, are introduced for the sake of completeness, but are placed within brackets.

In conclusion, the writer is glad to express his obligations to Prof. Calcott Reilly of Cooper's Hill, to the Rev. H. C. Watson and Mr. J. E. Pearson of Clifton College, and to Prof. H. S. Hele Shaw of University College, Bristol, for valuable aid of various kinds given during the progress of the work.

LONDON, 1*st March*, 1883.

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THE

STUDENT'S MECHANICS.

PART I.—FIRST PRINCIPLES.

§ 1. Introductory.

1. The Science of Mechanics has been studied so long and so successfully, that it has become what is called a *Deductive* Science. By this is meant that mechanicians have long arrived at certain general laws or principles, which have been so fully and so frequently confirmed, alike by observation, by experiment, and by the results of their application, that they take the very highest rank as Proved Scientific Truths. These principles being laid down as a basis, the whole of the science is deduced from them, and from the proper definitions of the terms employed, without any need of appeal to experiment or observation by the way. At the same time it must be pointed out, that all the results obtained may be verified, and have been verified, by actual observation and experiment; indeed, it is this accordance of the theoretical results, deduced from the principles, with the actual results obtained in practice, which forms, perhaps, the strongest evidence that the principles themselves are true.

2. Very few sciences, even of those which have been longest cultivated, have reached this stage of being *Deductive*. As an example of one which still remains *Inductive*, as it is called, we may take Chemistry. Here we have, no doubt, certain general laws established, such as that of atomic weights; but we can scarcely proceed a step under the guidance of this law without calling in experiment to help us on our way. For instance, if Chemistry were a deductive science, then, on learning the atomic

weights, and, perhaps, some other particulars, regarding any two chemical elements—say, carbon and iron—we ought at once to be able to describe all the compounds they can form, and the general character of those compounds. Every student of Chemistry, however, knows that this is not the case, and that the compounds of iron and carbon must be studied separately, as they occur. A book on Chemistry is, in fact, a collection of separate facts, which are ascertained by experiment, and which, though more or less *connected* by general laws, can by no means be *deduced* from them.

3. Of Deductive Sciences the most familiar example is Geometry, as given in the *Elements* of Euclid. There the whole of the work is based upon the axioms and postulates at the beginning, together with the definitions which are prefixed to each book. The same course is possible in Mechanics. Accordingly, the definitions and the axioms on which Mechanics is founded are here collected together in Part V.; and to these are subjoined certain "Laws," as they may be termed, which are themselves deductions from the definitions and axioms, but are so important and so frequently used, that it is well to bring them together in one place for ready reference.

4. There are, however, certain differences between the foundations of Geometry and those of Mechanics which must be noted. In the first place, to understand Euclid requires no knowledge beyond that of reading, writing, and arithmetic; and the axioms, definitions, &c., are supposed to embrace *all* that is needed for the proof of the various propositions. But to understand Mechanics, even in its most elementary form, it is necessary to be acquainted, not merely with Arithmetic, but also with the main facts, at least, of Geometry, Algebra, Algebraic Geometry (or Conic Sections), and Trigonometry.* As we advance farther in the subject, we find many propositions which are much more readily and simply proved by the aid of the Differential Calculus, and others which without it cannot be proved at all. Now, it is clearly needless to set forth at length the fundamental principles of these various branches of Mathematics, in order to complete the basis on which we found our fabric of Mechanics. All we need do is to give those

* *A few propositions* in the present work, which require a knowledge of *higher subjects*, are placed within brackets.

axioms, &c., which are necessary for Mechanics, but are not required in any of these introductory sciences, and with which the student, therefore, has now for the first time to become acquainted.

5. Again, the axioms of Euclid are some of them concerned with Geometry alone—*e.g.*, "Two straight lines cannot enclose a space;" while others have a much wider application—*e.g.*, "If equals be added to equals, the wholes are equal." The same is the case with the axioms of Mechanics. But, whereas in Euclid the latter class of axioms, such as the one quoted, are so familiar and universal that nobody disputes them, in Mechanics they are indeed wide and deep generalisations, but such as require much study thoroughly to grasp, both as to their nature and their evidence. Consequently, instead of appearing mere commonplaces, like those of Euclid, they will probably strike the student as something vast and uncertain, which he may be in doubt whether to accept or no. For this reason it is not sufficient to set them down baldly, and once for all, at the beginning of the book, as Euclid does. They must be to some extent explained and illustrated; and it must be shown that they are not extemporised, as it were, for the special behoof of this particular science, but that they are also true of all other cases in nature to which they can apply. Unless this can be done, these axioms are not entitled to take rank with the great generalisations on which Euclid builds his theorems.

6. Hence, the axioms of Mechanics are not here stated at the beginning, after the fashion of Euclid. They will be found dispersed through the book, and fitted with explanation and illustration wherever they first occur; and they are, finally, collected together at the end, where they can be referred to as required.

§ 2. Definition of the Science.

7. **Definition.**—*Mechanics is the Science of Motion and Force.*

8. Since Mechanics is thus concerned with two different things, *it is evident that it may* be divided into three branches, according

as we study each of these by itself, or both together. Accordingly, we have the following recognised branches of Mechanics :—

(1.) *Kinematics*, or the study of motion apart from force.

(2.) *Statics*, or the study of force apart from motion.

(3.) *Dynamics*, or the study of motion and force together.

9. In many books the two former branches are first developed singly, and then attention is given to the third branch. Although this may be logically correct, it does not appear to be well calculated to give a firm grasp of the subject from the beginning. Accordingly, in this treatise we shall begin by laying down the definitions of motion and force, and the fundamental laws of their action—in other words, the first principles of Dynamics. We shall then give separately the leading facts of Statics and Kinematics, as far as it is desirable to pursue them for our purposes; and, finally, we shall return to Dynamics, in order to give various higher theorems, which mostly require a knowledge of Kinematics and Statics for their proper comprehension.

§ 3. Definitions of Motion and Force.

10. Motion is so simple and familiar a condition that any definition in words would only confuse it. We may, however, consider how it is that we know a thing to be in motion. We then arrive at the following fact :—*A thing is known to be in motion, when it is continuously changing its position in space with reference to some other thing assumed to be fixed.*

11. Now if there were anything which we knew to be absolutely fixed in space, we might perceive absolute motion by change of place with reference to that thing. But as we know of no such thing, it follows that all motion, as tested and measured by us, must be relative—must relate, that is, to something which we assume to be fixed for the moment. Hence, the same thing may often be properly said to be at rest and in motion at the same time; for it may be at rest with regard to one thing, and in motion with regard to another. Thus, take the very homely instance of a man punting his barge up *a river*, by leaning against a pole which rests on the bottom, and by

advancing his feet successively on the deck as if walking. Such a man is in motion relatively to the barge; he is also in motion—but in a different manner—relatively to the current; he is at rest relatively to the part of the earth immediately under his feet; he is in motion relatively to the polar axis of the earth and to the sun; whilst it is easy to imagine a proper motion of the whole solar system such that he would be absolutely at rest in space.

12. Motion, then, as treated in Mechanics, is a condition of bodies in which they are continuously changing their position in space; and it is a *relative* condition, because this change is relative to some other body, which we agree to regard as fixed. This does not mean that there is no such thing as absolute motion, but only that we have no means of measuring absolute motion, or investigating it in any way, because we have nothing absolutely at rest by which to measure it; and therefore all the motions which we can investigate are relative motions.

13. **Definition.**—*A force is a physical cause of motion.*

14. This definition is only meant to give the sense in which the *term* "Force" is used in books on Mechanics. We have all of us an idea of Force as a thing, derived from our experiences in pushing, pulling, lifting, and doing other things, which, as we say, require *force*. In all such cases we know that we are either moving things, or at least tending to move them; and, therefore, our idea of the thing Force corresponds with our definition of the term Force. What this Force is in itself, however, is a much vexed question, which there is no necessity whatever to solve for the purposes of Mechanics. For those purposes it is sufficient to give the definition above, which is, in fact, to define Mechanics as the Science of Motions and of the Causes of Motion. It thus corresponds with all other branches of science which are in an advanced and satisfactory condition. For instance, Astronomy is the science of the motions of the heavenly bodies, and of the causes of those motions; Acoustics is the science of the phenomena of sound, and of the causes of those phenomena; and so forth. It may well be questioned whether branches of natural philosophy, which have not got beyond the mere description of phenomena,

and do not attempt to touch their causes, have any good right to be called "sciences" at all.

15. Something must be said about the other word in the definition—namely, Cause. This word it is impossible to avoid, since no other expresses the same meaning; but it has been greatly obscured by the immense quantity that has been written upon it; and, moreover, it is often used in different senses. For instance, if I were to ask the cause of the railway from London to Birmingham, I might be told that it was a number of persons joining together to find the funds necessary; or that it was the wish to facilitate communication between those places; or that it was the organised labour of a number of workmen; or that it was the mechanical action of the picks, shovels, and other tools which those workmen used. Now, it is in the last of these four senses that the word "cause" is used in our definition of force. A force is the immediate means of setting something in motion; in simpler language, it is whatever *makes something move;* just as the downward thrusting of the labourer's spade is what makes the earth be loosened in the railway cutting, and the subsequent lifting action is what makes it be deposited in the truck, to be carried away. It is this which is implied by the term "physical cause" in our definition.

16. From this idea of force as a cause we can at once deduce one important conclusion. We are quite familiar with the fact, that a cause may operate and yet may fail to produce its result, because it is counteracted by some equal and opposite cause. Thus, the writing of the words in this book might take place, and yet might produce no result if the paper, instead of being white, were the colour of the ink. Again, the lifting action of the labourer's spade might be exerted and yet produce no result, if the weight of earth were beyond his strength. In such cases, which are very common, we say that the cause *tends* to produce its result, but that it is counteracted, and so prevented from doing so. Applying this to force, we have the following axiom.

17. **Axiom.**—*A force always tends to produce motion, but may be prevented from actually producing it by the counteraction of an equal and opposite force.*

18. By most writers this fact is included as part of the definition of force; but it seems preferable to state it as an axiom, because it is virtually included in the definition as soon as we have defined force as a cause. For those, however, who adhere to the old view, no better form can be given to the definition of force than the following:—*Force is any cause which changes, or tends to change, a body's state of rest or motion.*

§ 4. Measurement of Force and Motion.

19. Having explained what we mean by motion and force, we must now go on to explain how they may be measured; for without tolerably exact measurement we can know nothing of any class of natural phenomena, except in a vague and superficial manner.

20. **Measurement of Motion.**—We have seen (Art. 11) that motion implies a continuous change of place with reference to something assumed as fixed. Hence, to measure motion we must measure the amount of this change of place. If we could make a mark at the spot which the moving body occupied at any instant, and another mark at the spot occupied at any subsequent instant, and afterwards measure the distance between these two marks (which we must suppose to be fixed), then that distance would clearly be a measure of the motion during the interval.

21. Thus, suppose the marks to be successive mile-posts on a railway, and that we have an express train and a goods train passing the first of these at the same moment. Then if we watched the two, and found that at the moment when the goods train passed the second mile-post, the express was passing the third, we should know at once that the speed of the express was twice that of the goods train. Hence, when the time is the same, the proper measure of motion is proportional to the distance passed through.

22. But it would not be necessary for us to go to the third mile-post at all. Suppose we watched by the second mile-post, and found that the express train arrived in two minutes after passing the first, and the goods train in four minutes after doing so, then we should know just as well that the speed of the first was twice

that of the second. We know this, because the time taken by the first to run a mile is one-half the time taken by the second. Hence, when the distance is the same, the proper measure of motion is inversely proportional to the time of traversing this distance.

23. Combining these two facts, it is evident that the measure of motion varies as the space described directly, and as the time of describing that space inversely. In other words, the speed, or the intensity of motion, is properly measured by the ratio of the space described, s, to the time, t, occupied in describing it; that is, by the fraction $\left(\frac{s}{t}\right)$.

24. It has, of course, been assumed here that the speeds are uniform throughout. If, for instance, the express train had been getting up its speed during the first mile, it might reach the second mile-post in two minutes, as before, but might then be travelling at a much higher speed than twice that of the goods train. In that case, assuming it to have now settled down to a uniform speed, we should have to wait till it reached the third mile-post before we could tell what that speed really was. Hence, we must in practice have two modes of measuring motion, according as it is uniform or variable.

25. If the motion is uniform, we simply adopt some convenient unit of time, say one second, and measure the speed by the distance, s, described in one second.

26. If the motion is variable, we may give as its measure at any instant the distance which it *would describe in one second*, if it continued during one second at the same value which it has at the instant considered. But it will be the same thing, if, instead of the space s, which is the space described in a unit of time, we take the value of the ratio $\frac{s}{t}$, where t is taken as a fraction of a second so small that during the time the speed does not appreciably vary, and s is the space described in time t. In the language of the Differential Calculus, this ratio is $\frac{ds}{dt}$.*

* When an excessively small part of any magnitude is considered, it is expressed by putting the letter d before the symbol expressing the magni-

27. In Mechanics the word velocity is used in place of speed. The foregoing may therefore be summed up as follows. When we say that one body has a greater velocity than another, we simply mean that it is passing through a greater distance than the other in the same time. But space and time are both capable of measurement. Hence, *if we can measure any interval of time, during which we know that the velocity remains constant, and can also measure the interval of space which a body travels over during that time, then the ratio of the space to the time, as thus given, is a proper measure of the velocity, in other words, of the motion, of the body.*

28. **Measurement of Force.**—A cause which is capable of producing an effect, but has not yet done so, is said to possess *power*. The amount of power which any cause possesses we can only measure by the amount of the effect which it is capable of producing. When it has produced the effect, there is no difficulty in measuring it. Thus, the explosive power possessed by an ounce of gunpowder is determined by the amount of explosive effect which it exercises if ignited. We have, however, no means of measuring directly the effect which a cause can produce, but has not produced as yet. All we can do is to measure the effect which a cause apparently similar in all respects (*e.g.*, another ounce of the same kind of gunpowder) has produced, and assume that the first cause possesses the power represented by that effect.

29. Now we have defined a force as a cause of motion. Hence, we see that, if a force has produced motion, it will be represented to us by the motion it has produced; and if it has not produced motion, it will be represented by the motion which has been produced by a force believed to be equal to it. But motion is measured in terms of velocity (Art. 23). Hence, other things being equal, forces are measured by the velocities which they cause or generate.

30. To show what is meant by the proviso "*other things being equal*," we may follow exactly the same course as in the case of velocity (Art. 21), and recur to the example of the express and goods trains. If force is measured by velocity, then, when the

tude, so that ds expresses an exceedingly small interval of space, and dt of time.

velocities are equal, the forces are equal also. Suppose, however, that the two trains, having started together, are running at exactly the same speed, but that the goods train is twice the weight of the express train; then we should not admit that the forces exerted by the two engines were equal, but should call that of the goods engine much greater than the other. Hence, we see that, if forces are to be measured by the velocities they generate, the *things they act upon* must be equal.

31. Again, suppose that the trains were of equal weight, but that the goods train reached a given speed from rest in one minute, while the express reached it in two; then again we should say that the goods engine had exerted a much greater force. Hence, we see that, if forces are to be measured by the velocities they generate, the *times during which they act* must be equal.

32. Thus we are led to the principle: *Forces are measured by the velocities they generate in the same or equal objects and in the same or equal times.*

33. The unit of force will, therefore, be that force which generates a unit of velocity in a unit of matter and in a unit of time. In England the unit of velocity is 1 foot per second, so that a unit of force is that force which in one second generates in a unit of matter a velocity of 1 foot per second. On the Continent, the corresponding velocity is 1 centimètre per second.

34. But we have already seen (Art. 16) that forces sometimes act without producing motion, being counteracted by opposite forces. How are we to measure them in this case? The obvious way of doing so is to find some known force, which always remains constant and unalterable, and to see how many units of this known force will exactly counteract or balance the force we are seeking to measure. Such a constant force is found in what we call *gravity*—in other words, the attractive force of the earth.

35. It is found that the amount of this attraction on any given piece of matter is the same at all times, and, practically, the same at all places, provided they are not removed from each other by a considerable part of the earth's circumference. The amount of *this attraction* is called the weight of the piece of matter. The *unit of weight in England* is the pound, and a pound is the weight

of a certain piece of metal preserved at the Standards Office in London. On the Continent, the unit of weight is the gramme.

36. Any particular force, then, may be measured by ascertaining the number of pounds weight which will counteract it. This we can do in various ways, according to circumstances. If the force is itself the weight of some body, we may measure it by putting the body into one scale of a balance and putting a number of pounds into the other scale, until the two balance exactly. In other cases we may make the force act through a string or rope to lift a certain number of pounds from the ground, or to extend a spring, the elongation of which, under known weights, has already been determined.

37. These two modes of measuring force—by the motion produced, or by the weight balanced—must be carefully borne in mind and distinguished from each other. As the first has to do with forces in motion, it is naturally called the Dynamical method of measurement, and as the other has to do with forces at rest, it is called the Statical method. They may be described as follows.

38. Forces are measured by the velocities which they cause or generate in the same or equal objects and in the same or equal times. This is the *Absolute* or *Dynamical* method, and forces when so measured are called *accelerating forces*.

39. Forces may also be measured by ascertaining the number of units of a standard force (in England, the number of pounds weight) which will exactly counteract them. This is the *Statical* method, and forces when so measured are called *statical forces*.

§ 5. Modes of Action of Force.

40. Forces may be supposed to act upon bodies in very many different ways. Forces might act apparently at random, and without any law whatever; but such forces cannot, of course, be studied in any way, and therefore we need not consider them further. All forces which are known in nature act according to definite laws; that is to say, they are connected with space and time by certain definite relations, which can be expressed in words or symbols.

41. First, as to relations to time.—*A force may either be a continuous force, that is, it may always be going on; or, it may be a discontinuous force, that is, may act at intervals only.* When a force is practically instantaneous in its action, and the actions occur at intervals more or less long, the force is called an Impulse. Examples of impulses are the blows of a hammer on an anvil, of a clapper upon a bell, &c. They occur at definite intervals, but the time they last is too small to be measured.

42. There may be other forces which are discontinuous but not impulsive; such is the action of a cam in machinery. A cam is a projection on the outside of a revolving wheel, which, each time that it comes round, lifts or moves some piece out of its way, then lets it fall back again. This lifting, however, is not an instantaneous, but a gradual action.

43. It will be observed that these cases of discontinuous forces are so only because they have been specially so arranged by the power of man. Their discontinuity is in fact apparent, not real. The forces of nature are, so far as we know, always continuous; that is, they never cease acting, although their action may be counteracted. They are also uniform; that is to say, if the relations of space remain unaltered, the force will remain always the same. In other words, if the force be expressed in symbols, the symbol for time does not enter into the expression.

44. Secondly, as to relations with space.—There is probably no limit to the number of ways in which forces may be supposed to act, and all kinds of suppositions are constantly made in the form of problems for the exercise of students. For instance, we may suppose that, at every point on a certain straight line, a person would find himself acted upon by a force of a definite amount, say 10 lbs., acting along that straight line; but that if he got away from the straight line the force would disappear. Again, we may suppose that at every point in a certain plane he would find himself acted upon by a force tending, we will say, to the northward, but that if he got off this plane the force would disappear. Or again, the force might not be constant, but the farther he got to the north the greater or the less might the force become. Or again, he might find himself always acted upon by a force which

tended to turn him in a certain direction, and then to keep him in that direction. Of this last we have an actual example in nature, in the force which causes a magnet, wherever it is, to turn north and south, and to keep pointing in that direction.

45. Yet again, we may suppose a person to be walking over the surface of a sphere, and find himself always drawn towards the centre of that sphere. This is what happens to all of us in passing over the surface of the earth. We are always being drawn towards the centre, and the force which draws us is sometimes called the "attraction of gravitation," and sometimes our own weight. But this is not all: for, if we rise above the surface, as in a balloon, or if we pass beneath it, as in a mine, we still find ourselves drawn towards the centre; and this will be the case however near to, or however far from, that centre we may arrive. Such a force is what is called a central force; it is as if a power existed at one particular point (in this case the centre of the earth) which was always pulling towards that centre every body that came within its range.

46. In other cases the force might always be pushing bodies away from the centre instead of towards it, which is expressed by saying that it is a repulsive, and not an attractive force. Again, the force may be the same in amount whatever the distance from the centre, or it may be different at different distances; and in the latter case, there may be any number of ways in which this variation of the amount with variation of the distance may take place. In the case of the earth, for instance, the amount varies inversely as the squares of the distances; by which is meant that if the distances are represented by 1, 2, 3, 4, &c., and if the amount of the force at distance 1 is represented by F, then the amounts at the other distances will be $\frac{F}{4}$, $\frac{F}{9}$, $\frac{F}{16}$, &c.

47. It is very desirable to master at once this conception of central forces; because, so far as we can see at present, the whole of the phenomena of nature are probably due to the action of central forces, and, therefore, their treatment has an importance infinitely beyond that of any other mode of force.

§ 6. Matter.

48. We have now defined force and motion, but it still remains to define that which force acts upon. The general name for this is *matter;* and many writers on Mechanics are content to assume that matter is a familiar object to all, and refuse to define it. Others give definitions. Of these, perhaps, the best is that of Thomson and Tait, who define matter as "that which can be perceived by the senses, or as that which can be acted upon by or can exert force." This last clause sets forth the very important truth, that matter both acts and is acted upon,—that it is not only the Object, but also the Source, of Force. For the purposes of Mechanics, however, a more precise definition is desirable, and this is now to be given.

49. **Definition.**—*Matter consists of a collection of centres of force distributed in space, and acting upon each other according to laws, which do not vary with time, but do vary with distance.*

50. This definition is of matter in general; but in practice we are always treating of some definite portions of matter, and we require names to express these portions, according to their size and other properties. The names usually employed are the following, beginning with the most elementary.

51. (1.) The centre of force itself is called an ultimate atom, or a physical point. Like a geometrical point, it has no assignable parts or magnitude, and cannot therefore be compressed, extended, or divided. It is, in fact, a geometrical point, conceived as having also the properties of exerting and receiving forces, and of being movable through space, whilst retaining its constitution unaltered. The word "point" is, perhaps, the simplest and shortest which can be applied to it; but it is sometimes desirable, for greater clearness, to use the word "centre."

52. (2.) A collection of points or centres acting on each other, and so intimately and closely bound up together that no known process or natural force can separate them, is called an *atom* or a *molecule.*

53. (3.) A collection of points, simply considered as so small

that for the purposes of any particular investigation, or for those of Elementary Mechanics in general, it may be considered as concentrated in a single central point, is called a *particle*. This word is used merely to imply that all questions of size, rotation, constitution, &c., are for the present left out of account.

54. (4.) A collection of points of any size or shape whatever, which for the purpose of any investigation is treated together as a whole, is called a *body*.

55. A few remarks may be subjoined. The above is a definition of Matter, only as a term of Mechanics and as concerned with Force. It does not assert or deny that Matter may have other properties — *e.g.*, the properties which distinguish the different chemical elements *may* be special properties unconnected with force. On the other hand, the definition does really embrace what are recognised as the general properties of Matter, such as Extension, Hardness, Colour, Temperature, &c. Of these Extension is included by saying that the centres are distributed in space, and the remainder are now recognised as properties depending upon Force. Again, the definition does not preclude the existence of different *kinds* of matter ; such, for instance, as that the particles of any one kind should only act on each other, and not on particles of the other kinds. Whether there is any such division into kinds in Nature is, however, doubtful, certainly it is not obvious or familiar ; and in this treatise it will be assumed that all the matter considered is of one kind, that the centres of force are all alike, and all act on each other by like laws.

56. There is one fundamental fact with regard to matter which must be stated, before we can even commence our investigations. It is expressed by saying that any given body, under the action of any given force, will only be caused to move over a finite distance in a finite time. In other words, the motions caused by any known forces will always be finite or limited motions.

57. The reason why we must make this limitation is obvious. Let us suppose there was a body which did not possess this property, and which, therefore, on being acted upon by any force, would assume a motion which was infinite. Then the first time such a *body was acted* upon by any force, it would be moved to

an infinite distance, and, therefore, we could never see or know anything concerning it. Everything we can measure or investigate in any way must itself be finite, and this applies to motion as much as to other things.

58. This fact is generally expressed by saying that matter has *Inertia;* and inertia is spoken of as a special property of matter. It is only so, however, as being a special case of the general fact that finite causes only produce finite effects. Retaining the name, the principle may be expressed as follows :—

59. **Principle of Inertia.**—*All things within our range of knowledge being finite, any known body under the action of any known force will only assume a finite motion in a finite time.*

60. **Measurement of Matter.**—Since matter consists of centres of force, which we assume to be all alike, *the quantity of matter in a body simply means the number of centres of force contained in it*, just as the quantity of shot in a given bag simply means the number of pellets. This might be called the absolute definition of Mass, mass meaning the quantity of matter in a body; and if we could actually count the number of centres of force, this would be a practical mode of measuring matter; but this is impossible, and we must find some other method. For this purpose we use the assumption that all centres of force act on each other alike, and with forces which are equal at equal distances (Art. 55).

61. Let us suppose that we can isolate a certain centre of force, and place it so that it is at practically equal distances from all the centres of force in a body. Then the force which the body exercises on this external point is simply the force due to one centre multiplied by the number of centres, and, therefore, is proportional to that number. Hence the force existing between the point and the body will be a measure of the body's mass. Now we cannot really isolate a single point in this way, but practically we can treat the earth as a standard isolated body, having the whole of its mass concentrated at the centre; and then we measure the mass of any other body by the force existing *under standard* circumstances between it and the earth. But this *force is called its* weight; and therefore we arrive at the principle

laid down by Newton from experiment, that *the mass of a body is to be measured by its weight.* We thus arrive at the following definition :—

Definition.—*The number of centres, or the quantity of matter, in a body is called its Mass; and it is measured by its weight.*

62. We have seen that the weight of a body is determined by comparing it with a standard unit of weight—*i.e.*, in England, the weight of the standard pound at or near London. In measuring mass, however, we need not make the latter restriction. Suppose we take the standard pound, or a copy of it, to any other place, and weigh it against a given body at that place: then, if the two weights are equal, we know that the number of centres of force in the two bodies are equal, since the total attraction exercised upon them by the earth at that place is the same. Hence the masses of the two are equal. In this way the standard pound is often looked upon rather as a standard of mass than of weight, since its mass is the same everywhere, while its weight depends to some extent on its position. We may even suppose the earth annihilated, or removed to an infinite distance. The weight of the pound would then become *nil;* but its mass would remain the same, and if it were made into a ball and fired out of a gun, it would fly no faster and no slower than it does now.

§ 7. Measurement of Force as related to Matter.

63. We have already seen (Art. 38) that forces are proportional to the velocities they generate in the same or like objects and in the same or like times. We are now in a position, having defined matter, to go farther, and see how forces may be measured when they cannot be applied to the same or like objects; in other words, how they are to be measured when the matter on which they act is taken into account.

64. We have seen that the solar system, considered mechanically, is a collection of like centres of force, or points, acting on each other by like laws, which vary with the distance. Let us now consider a *single point, as related to* any number, n, of other points, which are

all at the same distance from it, and form, in fact, an element or small part of the surface of a sphere, of which it is the centre. It is evident that the force existing between any one of these points and the first point will be the same; and therefore, the total force acting between the first point and the element will be proportional to the number, n, of points—*i.e.*, by the definition (Art. 61), to the mass of the element. If these forces are left to themselves, then, taking the central point as fixed, the element (since force causes motion) will move towards it or from it with ever-increasing velocity. But suppose another force to act directly upon the element and keep it at rest; then this force must balance the force existing between the element and the centre, and therefore it also must be proportional to the mass of the element. Hence we see that if the velocity is to be unaltered, force must vary as the mass it acts upon. But we have already seen (Art. 38) that when the mass is unaltered force varies as the velocity generated in a given time. Hence, by the ordinary law of proportion, when both are altered force varies as the mass acted upon multiplied by the velocity generated in a given time.

65. **Definition.**—*The product of the mass by the velocity of a body is called the Momentum; and forces considered in relation to the mass of the bodies they move are called Moving Forces.*

We may therefore state the conclusion at which we have arrived in Art. 64, by saying that moving forces are measured by the momenta which they generate in a given time. This is a principle laid down by Newton in Definition VIII. of the *Principia*, and in many later works announced as the Third Law of Motion.

66. On this system the proper, or absolute, mode of measuring force would be to isolate two centres of force, place them at the unit of distance from each other, and observe the relative velocity at the first instant after they began to move. This would form an absolute elementary unit of force; and the force acting between a centre of force and a body, in any other case, would be found by multiplying this absolute unit by the number of centres in the body—in other words, by its absolute mass—and by that function of the distance which expresses the law of the

forces. It is obvious that this process is impossible. As in the case of mass, we must resort to the standard furnished us by the earth on which we live. As already described (Art. 34), we begin by taking a certain piece of matter—such as the standard pound—which we agree to regard as a standard of mass. The weight of this pound, taken in London—since the force exercised by the earth may differ at different places—we take as the unit of weight. Any force, so long as it produces no motion, is measured by the number of such pounds which it will support—in other words, by the number of units of weight which it will balance. This is the measure of Statical Force, and its unit is the weight of a pound.

67. Again, if we consider forces as acting always on a unit of mass, and if we suppose that there is no force acting in the opposite direction, then these forces will be measured simply by the velocities which they generate in a given time. This is the measure of Accelerating Force, and its unit is a unit of velocity generated in a unit of time on a unit of mass. In England the unit of velocity is 1 foot per second, and since gravity, or the attraction of the earth, generates a velocity of about 32·2 feet per second in one second, we say that the value of the accelerating force of gravity is 32·2. This value we shall call g.

68. Thirdly, if forces act on different masses, and produce motion in them, then the forces are measured by the product of the mass and the velocity generated in a unit of time. This is the measure of Moving Force, and its unit is a velocity of 1 foot per second, generated in a mass of 1 lb. weight, and in a unit of time. Here, as before, we must consider that there are no forces acting in the opposite direction. In other words, the measure of moving force is only the measure of the *unbalanced part* of a force. The balanced part of a force is to be measured by statical standards only.

69. It must be clearly understood that, when we speak of statical, moving, or accelerating forces, we do not refer to different kinds of force, but only to forces as measured in different ways. The same force may take any one of the three names, according to the circumstances under which it acts.

§ 8. Laws of Motion.

70. We have now defined the things of which Mechanics is the science—viz., Motion, Force as the cause of motion, and Matter as that which exerts force and has force exerted upon it. We may now go on to lay down the leading principles of the science itself.

71. Newton, when he arrives at the same point in the *Principia*, proceeds by laying down three "Axioms or Laws of Motion," which, though not actually discovered by him, have ever since borne his name. These, literally translated from his own words, are as follows :—

"*First Law of Motion.*—Every body continues in its condition of rest, or of uniform motion in a straight line, except in so far as it is compelled by impressed forces to change its condition.

"*Second Law of Motion.*—Change of motion is proportional to the moving impressed force, and takes place along the straight line in which the force is impressed.

"*Third Law of Motion.*—Reaction is always opposite and equal to action ; or, the mutual actions of two bodies are always equal, and in the opposite directions."

72. These laws are usually given as independent axioms, merely adding proofs, or rather illustrations, drawn from experience. The laws, however, are not really independent. They can be derived from our definitions, by the help of certain axioms or principles which are not peculiar to Mechanics, but may be looked upon as universal truths of physical science. This we shall now proceed to show.

73. **Derivation of the First Law of Motion.**—This law can be deduced from a principle, which may be called the Principle of Conservation. It is best expressed by the simple words—"*Effects live.*" By this is meant that the effect of any physical cause does not die away or cease when the cause is withdrawn, any more than the life of an animal ceases when it is born ; nay, more, it will not cease at all, but will continue to live by its own vitality, as

it were, unless and until it is actually put an end to by some other cause of the opposite character. In a word, an effect does not cease of itself; it is only destroyed. And even when destroyed, it is not as though it had never been; for its destruction in itself produces an effect, and in some way an equivalent effect, on the agent which has destroyed it; and by the same law this effect also lives, unless and until it likewise is destroyed by some third agent, to which in turn it also communicates an equivalent effect; and so the generation is continued for ever.

74. The proof of the principle of conservation, like that of most other generalisations, lies mainly in the fact that the evidence in its favour is continually augmenting, while that against it is continually diminishing, as the progress of science reveals to us more and more of the workings of the universe. That it is true to some extent is shown by such every-day facts as that a stone continues to fly after it has left the hand, that waves continue to roll after the wind has dropped, that the horseshoe continues to glow after it has been withdrawn from the fire, and so forth. On the other hand, the apparent exceptions—*i.e.*, the cases in which effects seem to die away altogether, after a longer or shorter interval—are so many that it is not to be wondered at if for many ages the principle failed to impress itself on the human mind. But the progress of modern science has shown so many of the exceptions to be apparent only, not real, and has at the same time brought to light so many additional instances of the rule, that the current of thought has changed; and the danger is now lest men should follow the rule too blindly and implicitly, and extend it to regions where it has not been shown to hold.

75. Assuming the principle, the proof from it of the first Law of Motion is as follows. Motion is the effect of force. But effects live. Therefore, the particular effect called motion lives. Therefore, a body once acted on by a force, and so set in motion, will retain that motion unchanged—*i.e.*, the same both in intensity and direction—unless and until some other force supervenes to cause a change in it. But motion which is not changed in intensity, is uniform motion; and motion which is not changed in direction, is motion in a straight line. Hence, a body

not under the immediate action of any force, if it moves at all, will move uniformly in a straight line.

76. **Second Law of Motion.**—This law, as enunciated by Newton, is practically the same as the principle that force is measured by the velocity which it generates, and has already been proved in Art. 38. But Newton added to the law a very important scholium, or explanation, to the effect that any force acting on a body will produce its full proportionate effect of motion, whatever other forces or motions the body may be subjected to at the same time; "the new motion being added to the previous motion, if it is in the same direction, or subtracted from it, if it is in the opposite direction, or obliquely combined with it, if the directions are oblique, and compounded with it according to the directions of each." It is this corollary to the law which really introduces the new conception, and, accordingly, it has gradually been substituted for the law itself. The usual form in which the law stands is as follows:—*When any number of forces act upon a body,* at once, *each produces its whole effect in altering the magnitude and direction of the body's velocity as if it acted singly on the body at rest.*

77. As examples of this law various facts may be cited. If an object be dropped within a railway carriage in motion, it will fall perpendicularly on the floor, exactly as it would do if the carriage were at rest. If a bullet is fired from a rifle, it will fall to the earth exactly as fast as if it is dropped from the hand; for which reason elevation has to be given to rifles or cannon whenever the distance is great. Again, the surface of the earth, and everything on it, is always moving with great velocity round the axis joining the poles; but of this motion we are not even conscious, and it does not affect in the slightest degree the other forces in motion on the surface.

78. This law may be proved as follows. If a force, in the presence of any circumstances, fails to have its full effect in imparting motion, this can only happen for one of two reasons. Either the force does not act so as to produce its effect; or, although the force acts, yet the effect dies away again. But, by our definition of matter, forces are always acting, and if the distance be the same

are always the same; and by the principle of conservation effects live and do not die away. Hence, neither of the above suppositions, upon these principles, can be true; therefore, forces do act under all circumstances to produce their full effects, which is the proposition required.

79. **Third Law of Motion.**—Before we can understand this law, we must know what is meant by action, and how action is to be measured. Newton himself has not stated this explicitly, and hence there has been some confusion about the meaning of this law. In fact, if we are considering the action between bodies, that action must be measured in different ways, according as the bodies are at rest or in motion relatively to each other. In the first case, as Newton has elsewhere intimated, it is measured by the forces existing between the bodies, and in the second by the energy exerted, which we are not yet in a position to define. If, however, we confine ourselves to the ultimate elements of matter, as given by our definition, then there is no difficulty. For we have assumed matter to consist of centres of force, all alike, and all acting on each other according to the same laws. Hence, the force with which one centre acts upon a second is the same as that with which the second acts on the first, because the distance between them is the same whichever way it is measured. If we call either of these forces the action, the other will be the reaction; and it will be equal to it in magnitude, and opposite to it in direction. But this is what is asserted by the third Law of Motion.

§ 9. Fundamental Equations for Motion in one Straight Line and under a Single Force.

80. The laws of motion, with our definitions, enable us to lay down at once the fundamental propositions as to the action between two points or centres of force, or between two bodies each of which may be supposed to be concentrated in a single centre. Practical examples of the latter case are the attractions of the sun and earth, neglecting the disturbing forces of the planets, or the

fall of a body to the earth, neglecting the resistance of the air. In such cases the ideas involved become greatly simplified.

81. In the first place, since all motion is relative to some point assumed to be fixed, we shall naturally assume as our fixed point one of the two centres of force concerned, and thus investigate the motion of the other in reference to it. Thus, in the case of the earth and sun, we assume the centre of the sun as fixed; and in the case of a falling body, we assume the centre of the earth as fixed.

82. Secondly, if the body assumed to be in motion be a single centre of force, it is an absolute unit of mass; and if it be a group of centres, the distances of all of them from the stationary body are assumed to be the same, and therefore each centre forming the group will be acted on by the same force. The total force on the group is therefore always proportional to its number. Hence, the motion will be the same whatever be the mass; and therefore the idea of mass need not be included, at least as long as the two bodies are still at a considerable distance from each other. In other words, we have only to treat the forces as accelerating, not as moving forces. (Arts. 38 and 65.)

83. Thirdly, as no other forces are acting, the whole of the action will take place in the straight line joining the centres, and the question of the combination of forces, which has not yet been settled, does not enter.

84. Let us consider two bodies, A and B, and investigate the motion of A with regard to B, taken as fixed. The problem will be stated mathematically thus:—*Given the distance,* l, *between* A *and* B, *at the instant when* A *begins to move, and the accelerating force,* f, *with which* B *acts on* A; *to find the velocity of* A (*referred to* B) *at* t *seconds after that instant, and also the space,* s, *which it has passed over from its original position towards* B.

85. Let us, in the first place, assume that the force with which B acts on A is constant. This is, of course, against our definition of matter, and is never exactly true in nature. But it may be assumed as true—according to the ordinary principles of mathematical reasoning—when either the interval of space or the interval of time, which we are considering, is relatively exceedingly small.

The former of these suppositions holds practically in the case of falling bodies, where the variation of gravity due to the approach towards the centre of the earth may always be neglected. The latter supposition will enable us to calculate the effect of a varying force, by supposing it to be represented by the effect of an indefinite number of constant forces, each acting independently and for an indefinitely short interval of time.

86. We therefore begin by assuming that the force is constant. It will therefore be measured by the velocity f generated in one second (Art. 38). Moreover, by the second law of motion, this velocity will in no way affect the action of the force, which in the next second will again generate exactly the same velocity, f. At the same time, by the first law, the velocity, f, generated in the first second will remain unaltered. The velocity at the end of the second second will therefore be $f+f$, or $2f$. In the third second a third velocity, f, will have been added, making the velocity $3f$. By similar reasoning the velocity at the end of t seconds will be tf; or, if V be the velocity of A at the end of t seconds, we have

$$V = ft \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

87. The above equation has been proved only when t is an integral number of seconds. It remains to show that it holds for any value of t. For this purpose we may suppose the force to act by a number of successive impulses given at equal small intervals, dt, and each generating the same small velocity, v. Then, just as above, the actual velocity at the end of any number, n, of such intervals will be the sum of the n velocities generated; or, if V be this velocity, $V = nv$. Now, let n be the number of intervals, dt, which make up one second, so that $n\,dt = 1$. In this case the velocity generated is the acceleration. Then $V = f$; hence the equation, $V = nv$, becomes $f = nv = \frac{v}{dt}$, or $v = f\,dt$.

Again, let n be the number of intervals making up any other time, t, so that $n\,dt = t$. Then the equation becomes

$$V = nv = nf\,dt = f \times n\,dt = ft.$$

Since the interval dt may be any whatever, this equation is quite

general: and we may therefore take as the general relation between accelerating force and velocity

$$V = ft.$$

88. We have now to find the space, s, described in the time t. For this purpose we adopt the same artifice as before. Suppose the time, t, to be divided into a very large number, n, of very small equal intervals, dt. Then we may suppose that during each of these intervals the velocity remains constant at the value it has at the beginning of the interval; and the smaller the intervals, and therefore the larger their number, the nearer will this supposition be to the actual truth. But if the velocity be constant, the space described in each interval will be simply the velocity × the interval. Now, by equation (1), the velocities at the successive intervals, 1 to n, are $0, f \times dt, f \times 2\,dt, f \times 3\,dt, \ldots . f \times (n-1)\,dt$. Hence the spaces described in the successive intervals are 0, $f \times dt^2$, $2f \times dt^2$, $3f \times dt^2, \ldots . (n-1) f \times dt^2$. But the total space must be the sum of the spaces described in the successive intervals, and it is therefore equal to $f\,dt^2\,[1 + 2 + \ldots . + (n-1)]$, or to $f\,dt^2 \times \frac{(n-1)\,n}{2}$. But $n\,dt = t$; hence this may be written

$$\frac{ft^2}{2} - \frac{ft^2}{2n}.$$

The larger n is, the nearer will this expression be to the truth; but the larger n is, the smaller does the second term of the expression become. Hence, in the limit this second term will vanish, and the true value of the space described in the time t is $\frac{ft^2}{2}$.

[For those acquainted with the Integral Calculus this proposition can be put more briefly and clearly. The total space described is clearly the sum of the indefinitely small spaces described during each of the intervals dt. But for each of these spaces we may suppose the velocity constant, and therefore the space described is the velocity at that instant multiplied by the length of the interval. This is expressed symbolically by

$$ds = v\,dt.$$

But $v = ft$. Hence

$$s = \int ds = \int ft\, dt = \frac{ft^2}{2}.\]$$

89. In the above investigation we have started with the instant when A began to move, so that A's initial velocity is 0. Let us, however, suppose that A has an initial velocity, v_1, in the direction of B. Then, by the second law, this velocity will not in any way be affected by the action of the force, f, acting in the same direction. Hence the velocity at the end of the time t will be

$$V = v_1 + ft,$$

and the space described in the same time will be

$$s = v_1 t + \tfrac{1}{2} f t^2,$$

$v_1 t$ being evidently the space described by the point in the time t with the velocity v.

If the velocity is in the opposite direction, it will be diminished by the action of the force f, and the corresponding equations will be

$$V = v_1 - ft$$

$$s = v_1 t - \frac{ft^2}{2}.$$

90. Lastly, to find the relation between velocity and space described. If A start from rest, then, since $v = ft$, and $s = \frac{ft^2}{2}$, we have

$$v^2 = f^2 t^2 = 2fs.$$

If A have an initial velocity, v_1, then

$$v = v_1 \pm ft \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)},$$

$$s = v_1 t \pm \frac{ft^2}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}.$$

Hence

$$v^2 = (v_1 \pm ft)^2 = v_1^2 \pm 2f\left(v_1 t \pm \frac{ft^2}{2}\right).$$

Therefore

$$v^2 = v_1^2 \pm 2fs \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{(C.)}$$

This last is the general form of the equation connecting velocity and space described.

91. The equations (A), (B), and (C), are the general equations for the motion of a body in one straight line, and under the action of a single accelerating force; and they may be considered as fundamental equations for Dynamics. By Art. 68 they will also apply to the case of moving forces; for if P be the force, m the mass moved, and f the velocity generated in a unit of time, on the acceleration, we have by that article,

$$P = mf, \text{ or } \frac{P}{m} = f.$$

This gives the value of the acceleration in this case, and the same proof will then apply; we may therefore use the above equations, writing $\frac{P}{m}$ for f.

§ 10. Composition of Forces.

92. **Representation of Forces by Straight Lines.**—Many problems in Mechanics can be readily solved by geometrical methods, thus dispensing with the need of the higher calculus. This is due to the fact that forces can be completely represented, for mechanical purposes, by straight lines. Thus, if we take the absolute mode of measuring a force, it is represented by the velocity which it will generate in a unit of time and in a unit of mass. Now, this velocity has a certain value, which is measured by the number of feet moved through per second; and it has a certain direction, which is measured by the angle it makes with some line taken as a fixed standard within the plane in which it moves. For example, the direction in which a *ship is* steering is given by the angle it makes with the meridian, *or the line of* north and south. But direction and number can

both be represented geometrically. Assume any point, O, Fig. 1, in the plane of the paper to be the material point or centre on which the force acts. Draw any line, O N, in this plane to represent the direction taken as the standard—for instance, north and south. Draw through the point O a line O P, making with this line the same angle which the direction of the force makes with the standard line; and along O P measure O A, having the same number of units of length as the velocity has. These units of length may be actually the same—*e.g.*, a velocity of 2 feet per second may be represented by a line 2 feet long; or they may be on a different scale—*e.g.*, we may represent the velocity of 2 feet by a line 2 inches long, which is said to be on a scale of $\frac{1}{12}$. In either case the line O P completely represents in magnitude and direction—that is, in its only essential properties—the given force; and if we represent other forces by similar lines to the same scale, then any geometrical proposition we can prove to hold of these lines will also hold of the forces they represent.

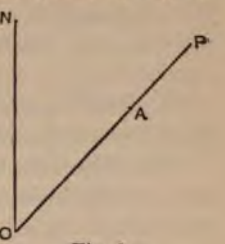

Fig. 1.

Or again, we may take the statical definition of force, and say that it is fully determined when we know its point of application, its direction, and the number of pounds pressure it will balance. If, then, we draw O P from the point of application, and make O A represent to any convenient scale the number of pounds pressure in any given case, then O A will completely represent the force in that case.

93. Having established the fact that forces may be represented, for the purposes of Mechanics, by straight lines, we are now in a position to go on and consider what will happen when not one, but several forces act together upon the same point. The fundamental problem in this department is to find the resultant of any given set of forces acting together upon a point, where the word resultant is used in the sense of the following definition :—

94. **Definition.**—*When the effect of* several forces acting together

upon a point is found to be the same in all respects as that of a single force supposed to act by itself on the point in their place, then this supposed single force is called the Resultant of the set of forces.

95. **Theorem.**—*There must always be a single resultant for any set of forces acting on a point.*

Either the point begins to move (or alters its motion) under the action of the forces, or it does not. If it moves, it can only move in one definite direction, and with one definite velocity; and by the definition of force, this effect is the same as if it were caused to move by the action of a single force tending in that particular direction, and of such amount as to generate that particular velocity. If it does not move, then the effect, by the first Law of Motion, is the same as if the point were not acted on by any forces at all; so that the resultant of the forces is zero. There is then equilibrium, according to the following Definition :—

Definition.—*When the resultant of the forces acting upon a point is zero, there is said to be equilibrium between the forces, and the point is said to be in equilibrium.*

96. **Problem.**—*To find the resultant of two forces acting on a point along the same straight line and in the same direction.*

By the scholium to the second law (Art. 76), or by the principle of conservation, which includes it, each force will produce its full effect independently of the other. Hence, if A and B are the absolute values of the forces, the point will have imparted to it a velocity of A feet per second, and also a velocity of B feet per second, both in the same direction. It will therefore have a velocity of $A + B$ feet per second in that direction. This is the same effect as if it had been acted on by a single force whose measure was $A + B$. Therefore the resultant in this case is the sum of the forces.

97. **Problem.**—*To find the resultant of two forces acting on a point along the same straight line, but in opposite directions.*

As before, each force will produce its effect independently of the other. Let A and $-B$ be the forces (the negative sign indicating the opposition in direction, as in Algebraical Geometry), and

let A be the greater and = B + C. Then, by the last theorem, the effect of A is the same as the effect of the two forces B and C, of which it may be considered the resultant. Hence, we may treat the centre as being under the influence of three forces—viz., two equal forces, B and − B, acting in opposite directions, and a third force, C, acting in the same direction as B. But, by the principle of symmetry, the effect of the two forces B and − B must be zero. Hence, the whole effect is that due to the third force C, or A − B, and the point moves as if it were acted on by a single force = A − B. Therefore, in this case the resultant is the difference of the two forces.

98. What we have here called the principle of symmetry is very widely used in Applied Mathematics, but is generally assumed without any discussion. As this is the first time that we have introduced it, and as we shall use it in other cases, it is desirable to give some account of it.

99. The principle, in its general form, may perhaps be best stated thus :—*When a cause, or set of causes, is so related to two opposite effects that there is no reason whatever why one of those effects should take place rather than the other, then neither of the effects will be produced by the cause or causes; and this relation is said to be a relation of symmetry.*

100. Of this principle, as of others, it may be true that when thus stated in its general form it is not easy at once to grasp its bearings. An illustration or two will make it clearer. We put a rein or a curb on each side of a horse's mouth, and then by pulling on both together we know that we shall not cause him to turn to either side, because there is as much reason to turn to the one as to the other. In Euclid, the axiom that figures which coincide are equal to each other, really rests on this principle, since there is no reason why one of two such figures should measure more than the other, or why it should measure less. In ordinary mechanical practice we admit at once that two equal weights suspended over a pulley by a weightless string will remain at rest, because there is no reason why either should either rise or fall; and for the same reason that two equal weights suspended from the ends of an equal-armed horizontal lever will also remain

at rest. This last fact has, indeed, been made the basis of a complete system of mechanics, which would thus rest directly on the principle of symmetry.

101. **Problem.**—*To find the resultant of any number of forces acting on one point and in one straight line.*

Let the positive forces (or those which tend in the positive direction) be A, B, C, &c., and let the negative forces be $-a, -b, -c,$ &c. Then by Art. 96, the resultant of the forces A and B is a force $(A+B)$; the resultant of the forces $(A+B)$ and C is a force $(A+B+C)$, and so on; therefore the resultant of the positive forces is $(A+B+C+\ldots)$. Similarly, the resultant of the negative forces is $-(a+b+c+\ldots)$. And by Art. 97, the resultant of these two forces acting in the same line, but in opposite directions, is given by taking their difference, or by $(A+B+C+\ldots) -(a+b+c+\ldots)$. Hence we have the law that *the resultant of any number of forces acting in one straight line is found by taking the algebraical sum of the forces.*

102. We have now to consider the question of finding the resultant of forces which do not act in the same straight line. This we shall proceed to solve by the aid of the principles already laid down, including that of symmetry (Art. 99). We shall begin with the simplest case, that of two forces whose directions are at right angles.

103. **Problem.**—*A point or centre is acted on by two forces whose directions are at right angles; to find the resulting motion of the point.*

Suppose the two forces to begin to act on the centre at rest, and to act upon it for an indefinitely small time, dt, and then to cease. If we assume, for the moment, that only one force has acted, it will have generated in that time a certain velocity, $f\,dt$ (Art. 87), where f is the measure of the force, being the velocity which would be generated if the force continued constant for one second. Since the time considered is indefinitely short, we may consider the force as constant during that time.

This velocity, $f\,dt$, will cause the centre to describe, in the next *element of time,* a certain indefinitely small space, ds, propor-

tional to the velocity. Hence, ds is proportional to f. Let P (Fig. 2), be the position of the centre at the end of the first interval, dt, and let P Q be the element of space which would be described in the second interval. Then P Q is proportional to f. Let P R be the element of space which would be described in the second interval, supposing the other force, f_1, to be the only one acting. Then by similar reasoning P R is proportional to f_1. And by hypothesis P R is at right angles to P Q.

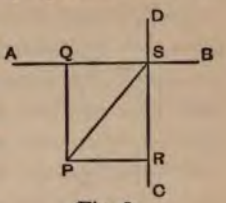

Fig. 2.

We have now to find the space actually described by P under the joint action of the two forces. By the second law (Art. 76), f will still produce its full effect, and will therefore actually cause P to describe the space P Q, except in so far as that space is increased or diminished — in other words, in so far as P is accelerated or retarded—by the action of f_1. But since the direction of f_1 is at right angles to P Q, it has no more tendency to produce an acceleration of P along P Q than to produce a retardation, and *vice versâ*—in other words, f_1 is symmetrical with respect to motion along P Q, and therefore, by the principle of symmetry, it will produce no effect either in accelerating or retarding. Therefore P will still travel the full distance P Q in that direction.

But by exactly similar reasoning, it will also travel the full distance P R in that direction; for by the same principle of symmetry, the fact that it is also moving in the direction P Q will have no effect on its motion in the direction P R.

This amounts to saying that, at the end of the time dt, P will be found somewhere on the line, A Q B, which is drawn through Q at right angles to P Q; and also somewhere on the line C R D, which is drawn through R at right angles to P R. Hence its actual position must be the point S, in which these two lines meet. And since the elements of space here considered are indefinitely small, the path of the centre will not differ from a straight line joining its extreme positions; in other words, from the straight line P S.

But P S is *the diagonal* of a rectangle, the sides of which are

proportional to f and f_1. Hence, P S is similarly proportional to $\sqrt{f^2+f_1^2}$; and, therefore, the path of the particle is exactly the same as if it had been acted on during the first interval, dt, by a force $\sqrt{f^2+f_1^2}$, whose direction coincided with P S. Hence, we have this result:—*When a centre is acted upon by two forces at right angles to each other, the effect is exactly the same as if it were acted upon by a single force, which is represented in magnitude and direction by the diagonal of any rectangle whose sides represent in magnitude and direction the two forces acting.*

In the foregoing proof we have supposed, for simplicity, that the forces cease to act during the second interval; but by the second law of motion the action of the forces during that interval will not alter the effect due to the action of the forces during the first interval, but must simply be added to or subtracted from it. But, by similar reasoning, we may show that the effect of this action in the second interval will be the same as if the diagonal force $\sqrt{f^2+f_1^2}$ continued to act during that interval; and therefore the result which we have arrived at will hold also for the second interval, and also for the third, fourth, fifth, &c., and is generally true for any time during which the two forces continue to act on the centre.

104. **Corollary.**—The above proof gives the principle of the combining, or, as it is usually called, the compounding of two forces at right angles to each other, which act together on any centre. Conversely, if a centre is acted on by a single force, P S, we may in thought regard it as under the action of two forces in any two directions, P Q, P R, at right angles to each other, provided we consider the values of these forces to be represented by P Q, P R, the sides of the rectangle of which P S is the diagonal. On this supposition we are said to resolve the single force into two component forces at right angles to each other.

105. We have now to extend the proposition to cases where the forces are not at right angles.

106. **Parallelogram of Forces.—Problem:**—*A point is acted on by two forces whose directions are at any angle with each other; to find the resulting motion of the point.*

Precisely as in the last proposition, we may represent the effect of the two forces f and f_1, considered singly, by two straight lines, P Q, P R (Fig. 3), drawn from the position of the point at the end of the first interval, $d t$. Complete the parellelogram P Q S R, and join P S. Draw Q A, R B, perpendicular to P S, and complete the parallelograms C Q A P, D R B P. Then, by Art. 104, we may consider the force P Q as resolved into the two forces, P C, P A, at right angles to each other, and the force P R as resolved into the two forces, P B, P D, at right angles to each other. But by geometry it is evident that A Q = B R, and therefore P C = P D; hence the force represented by P C is equal in magnitude and opposite in direction to the force represented by P D; and therefore, by the principle of symmetry, they will cancel each other, and will have no effect upon the motion of P. Hence the motion of P will be entirely due to the forces represented by P B and P A, and therefore it will be along the common direction of those forces, that is, along the line P S.

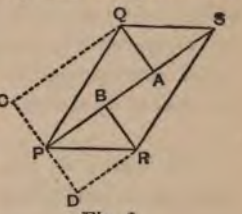

Fig. 3.

Moreover, by geometry, P B = A S; therefore P A = B A + A S = B S, and therefore P B + P A = P S. Hence the joint effect of the forces f and f_1 will be to make the centre, P, move through the space, P S; which is the same as the effect of a single force represented by P S.

Hence, just as before, we establish the general proposition:—*If a point is acted upon by two forces at once, whose directions make any angle with each other, the effect will be the same as if it was acted on by a single force, represented in magnitude and direction by the diagonal of the parallelogram, whose sides represent in magnitude and direction the two forces acting.*

107. **Corollary.**—Conversely, if a point be acted upon by a single force, P S, we may in thought regard it as acted upon by two forces in any two directions, P Q, P R, provided we consider these forces to be represented by P Q, P R, the sides of a parallelogram, of which P S is the diagonal.

108. *We have thus* established the proposition of the paral-

lelogram of forces, which forms practically the foundation of Statics. It should be observed that the forces, in the above proof, have been represented by the velocities generated by them, as in dynamics, and not by the pounds weight they will balance, as in statics; but it is clear that a proposition which is true of things when measured in one way cannot become false when they are measured in another way, and therefore the proposition may be at once assumed to be true for the science of statics as well as for that of dynamics.

109. Let P Q, P R (Fig. 3) represent two forces, Q and R, acting on a point, and let P S, the diagonal of the parallelogram, represent the force S, which, as shown above, is their equivalent. Then S is the resultant of P and Q, and P and Q are said to be the components of S in the directions P Q, P R respectively. The forces Q and R are said to be compounded in the direction P S; and the force S is said to be resolved in the directions P Q and P R.

110. **Problem.** — *To find the numerical relation between the resultant of two component forces, and either of the components.*

This means simply to find an expression for the diagonal of a parallelogram in terms of its sides. Let P Q S R (Fig 3) be the parallelogram, in which the angles Q P S, R P S are supposed to be known. Then, by Trigonometry,

$$P S = P Q \times \frac{\sin Q P R}{\sin R P S}$$

$$P S = P R \times \frac{\sin Q P R}{\sin Q P S}.$$

If Q P R be a right angle, as in Fig 2, these equations become

$$P Q = P S \times \sin R P S = P S \times \cos Q P S$$
$$P R = P S \times \sin Q P S = P S \times \cos R P S.$$

111. The above proof of the parallelogram of forces is not the *same as has* been given in most works on Mechanics. The proof *usually given is* known as Duchayla's; and as the importance of

the proposition makes it desirable that the student should examine it from various points of view, we shall proceed to give this proof also.

112. The proof rests on a new principle, which has been called, *The principle of the transmission of force through a rigid body.* It is enunciated as follows:—(Axiom)—*A force may be supposed to act at any point in its direction, provided that point be considered as rigidly attached to the point or centre on which the force really acts.*

113. Thus, let P (Fig. 4) be a force acting on a point or centre at A, and let B be a point in P's line of action. Then, if we suppose B to be rigidly connected with A, so that the distance between them is always unalterable, the effect on A will be the same whether we suppose the force to act at A or at B.

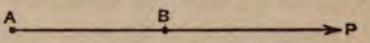

Fig. 4.

114. With regard to this principle, it should be remarked that the point B, and the connection, of whatever kind, between it and A, must be considered to possess no mass: because the effect of the same force cannot remain the same if the mass on which it acts is increased. In reality, the principle applies to the action of force on finite bodies, not on single points: and accordingly it has been enunciated as follows:—*If a force act upon a rigid body, it may be supposed to act at any point in its line of direction, without any alteration taking place in its effect.*

115. Thus, if we examine the force required to lift a ball of lead by a string, we shall find it to be exactly the same, whether the string is attached to the top of the ball, or whether it passes through a vertical hole made in it, and is attached at the bottom or anywhere between the top and bottom. In this way the principle may be proved experimentally; but it must be owned the instances of its truth are not very clear or familiar, because other circumstances almost always interfere. Its truth will scarcely be fully apparent, until the effect of a force of any kind upon a rigid body has been examined; and this belongs to a later stage of the subject. Assuming, however, that the principle is true, we may use it to prove the parallelogram of forces, as follows.

116. **Problem.**—*To prove the parallelogram of forces so far as the direction of the resultant is concerned.*

When the forces are *equal*, it follows from the principle of symmetry that the direction of the resultant will bisect the angle between the directions of the forces; hence, if we represent the forces in direction and magnitude by two equal lines drawn from the point at which they act, the diagonal of the parallelogram described upon these lines will be the direction of the resultant.

Next, suppose that the proposition, just proved for equal forces, is true for two unequal forces, P and Q, and also for P and R; we shall show that it will be true for P and Q + R.

Let A (Fig. 5) be the point of application of the forces; take A B to represent P in direction and magnitude, and A C to represent Q; complete the parallelogram A B D C, then by hypothesis A D is the direction of the resultant of P and Q; and, since a force may be supposed to act at any point of its direction, we may consider D as the point of application of the resultant of P and Q; or we may suppose the forces, P and Q, themselves to act at D, P being parallel to A B and Q to A C; or still further, we may suppose P to act at C, instead of D, in the direction C D.

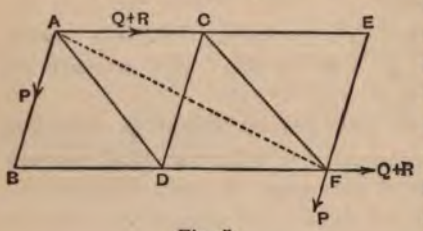

Fig. 5.

Again, the force R which acts at A may be supposed to act at C; take C E to represent it in direction and magnitude, and complete the parallelogram C D F E. Then by hypothesis C F is the direction of the resultant of P and R, taking both to act at C: hence the resultant of P and R may be supposed to act at F, or P and R may themselves be supposed to act at that point parallel to their original directions.

Lastly, the force Q, which is at present supposed to be acting at D in the direction D F, may be supposed to act at F.

Hence we have reduced the forces P and Q + R acting at A, to *P and Q + R acting* in the same directions at F; consequently F is

a point in the line of action of the resultant, and therefore A F is the direction of the resultant: that is, if the proposition be true for P and Q, and also for P and R, it is true for P and Q + R.

But the proposition is true for P and P, and also for P and P, therefore it is true for P and 2P, therefore for P and 3P, and so on; therefore generally for P and mP.

In like manner the proposition may be extended to mP and nP (m and n being whole numbers), that is, to any *commensurable forces.**

The proposition is extended to incommensurable forces as follows:—

Let A B, A C (Fig. 6), represent any two incommensurable forces; complete the parallelogram A B D C, and if A D is not the direction of the resultant, let it be A E. Suppose A C to be divided into a number of equal parts, each part being less than E D, and suppose distances of the same magnitude to be set off along C D beginning at C, then one of the divisions must fall between E and D; let F be the point which marks the division, and complete the parallelogram A G F C; then A F is the direction of the resultant of the *commensurable* forces A G, A C. But A F makes a larger angle with A C than A E, that is, the resultant of A G and A C lies farther away from A C than the resultant of A B and A C, although A G is less than A B, which is absurd: hence A E is not in the direction of the resultant. It may be shown in like manner that no other line is in that direction except A D. Hence the proposition, which was true for *commensurable* forces, is true for *incommensurable* forces also.

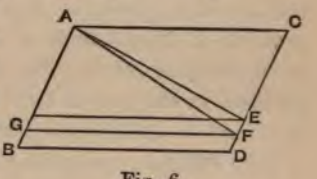

Fig. 6.

117. *To prove the parallelogram of forces with respect to the magnitude of the resultant.*

Let A B, A C (Fig. 7), represent the forces; complete the

* Two quantities are said to be *commensurable* when their ratio can be expressed *by the ratio of two whole numbers.*

parallelogram A B D C, join D A and produce it to D′, making A D′ *equal* to the resultant of A B and A C *in magnitude;* complete the parallelogram A B C′ D′, and join AC′.

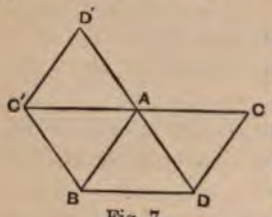

Fig. 7.

Then, since A D′ is equal to the resultant of A B and A C, and drawn in the direction opposite to that of their resultant, the three forces A B, A C, A D′ will balance each other, and therefore any one of them is in the direction of the resultant of the other two; hence A C is in the direction of the resultant of A B, A D′; but A C′ is also in that direction, therefore A C, A C′ are in the same straight line. Hence A D B C′ is a parallelogram; therefore A D = B C′: but B C′ = A D′, therefore A D = A D′. And by construction A D′ represents the resultant of A B and A C in magnitude, therefore A D also represents the resultant, and the proposition enunciated is true.

118. The relations between two component forces and their resultants, as given in Art. 110, enable us to solve the general problem of the resultant of any number of forces acting together upon the same point. For simplicity, we shall first consider the problem in the case of two dimensions—in other words, when all the forces lie in the same plane.

119. **Problem.**—*Any number of forces act at the same point, their directions all lying in the same plane: to find the direction and magnitude of their resultant.*

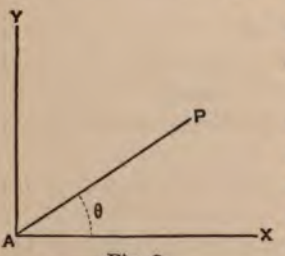

Fig. 8.

Let P be any one of the forces acting at the point A (Fig 8). Let the plane of the paper be that in which the forces act; in that plane choose any two lines at right angles to each other, A X and A Y, and let θ be the angle which the direction of P makes with A X.

Then P *is* equivalent to

$P \cos \theta$ acting in the direction A X,
together with $P \sin \theta$ „ „ A Y.

In like manner, a force P′, the direction of which makes an angle θ' with A X, is equivalent to

$P' \cos \theta'$ acting in the direction A X,
together with $P' \sin \theta'$ „ „ A Y.

And so on for any number of forces. Hence, adding together the forces which act in the same direction, we shall have a system of forces $P \cos \theta$, $P' \cos \theta' \ldots$, acting along the line A X ; and by Art. 110 this is equivalent to

$P \cos \theta + P' \cos \theta' + \ldots$ acting in the direction A X.*
Similarly we have $P \sin \theta + P' \sin \theta' + \ldots$ in the direction A Y.

For shortness' sake, let

$$P \cos \theta + P' \cos \theta' + \ldots = X,$$
$$P \sin \theta + P' \sin \theta' + \ldots = Y,$$

and let R be the required resultant, ϕ the angle which its direction makes with the line A X ; then

$$R \cos \phi = X,$$
$$R \sin \phi = Y ;$$
$$\therefore \tan \phi = \frac{Y}{X}, \quad R^2 = X^2 + Y^2.$$

These formulæ determine the direction and magnitude of the resultant of the system of forces.

[120. Next, let us consider the general problem in three dimensions, in which there is no limitation whatever as to the forces. The enunciation will be as follows :—

121. **Problem.**—*Any number of forces act upon the same point; to find the direction and magnitude of their resultant.*

Let A be the point (Fig. 9), and take any three axes, A X, A Y,

* If any of the forces act in the opposite direction to A X, they must be *given the negative sign.*

AZ, at right angles to each other, to which the forces may be referred. Let AP be any one of the forces P, and let x, y, z be the angles it makes with the axes: draw PM perpendicular to AZ, and PN perpendicular to the plane YAX, and therefore parallel to AZ. Then PM is at right angles to PN, and PM, PN are the components of P in the directions AN, AZ respectively. Again,

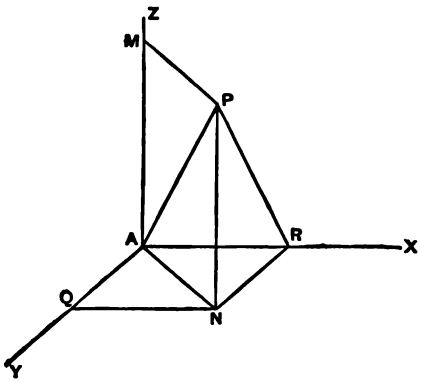

Fig. 9.

if we draw NR, NO, perpendicular to AX and AY respectively, then NR and NQ (or AQ and AR) represent the components of AN in the directions AY and AX. Thus on the whole we have AP represented by the three components AR, AQ, AM. But AR is at right angles both to PN and NR: therefore it is at right angles to the plane PNR, and therefore to the line PR, which meets it in that plane. Hence, since ARP is a right angle, $AR = AP \times \cos x$.

By similar reasoning $AQ = AP \times \cos y$.

And by construction $AM = AP \times \cos z$.

Hence, *any force, P, is always equivalent to three forces acting parallel to any three rectangular axes, and having the values, P cos* x, *P cos* y, *P cos* z *respectively, where* x, y, z, *are the angles which the direction of P makes with the axes respectively.*

Hence, as in the last article, the whole system will be equivalent to the following:—

$$P\cos x + P_1\cos x_1 + \ldots = X \text{ along the axis of } x.$$
$$P\cos y + P_1\cos y_1 + \ldots = Y \quad \text{,,} \quad \text{,,} \quad y.$$
$$P\cos z + P_1\cos z_1 + \ldots = Z \quad \text{,,} \quad \text{,,} \quad z.$$

And if R be the resultant, we shall have

$$R \cos (RX) = X,$$
$$R \cos (RY) = Y,$$
$$R \cos (RZ) = Z,$$

where RX is the angle which R makes with the axis of x, &c. We also have

$$R^2 = X^2 + Y^2 + Z^2.]$$

122. **Parallelogram of Velocities.**—From the fact mentioned in Art. 108, that in the proof of the Parallelogram of Forces the forces have been represented by the velocities generated by them, it follows at once that the velocity of a point at any moment may be supposed to be resolved into two component velocities, on exactly the same conditions as a force may be resolved—viz., that the lines representing the components form the two sides of a parallelogram, of which the diagonal represents the actual velocity.

This proposition is usually demonstrated independently as the Parallelogram of Velocities, and is often stated in the converse form—namely, that two velocities, existing at the same time, may be compounded into a resultant velocity. This, however, is a confusion of ideas. It is quite right to regard a body as under the action of two forces at the same time, and to compound their effects, because such forces may be separate realities; but the velocity of a body at any instant must be in one definite direction, and of one definite amount, and it is only in thought that it is possible to analyse it into two velocities tending in two different directions, and capable of being studied independently.

123. **Resolution and Composition of Velocities.**—Exactly as in Art. 121, it may be shown that any velocity, V, of a particle in space may be resolved into three velocities along any three rectangular axes that may be chosen, the values of the component velocities being

$$V \cos x,\ V \cos y,\ V \cos z,$$

respectively.

§ 11. Moving Forces—Energy—Work.

124. In this section we propose to examine the leading principles *with regard to* Moving Forces, as defined Art. 65.

This investigation differs from that of Accelerating Forces, already discussed in § 9, in two main particulars.

125. Firstly, instead of confining ourselves to the action of two centres of force on each other (or of two particles which may be treated as collected in two centres, and as equal in all respects), we have here to treat of the action of a centre of force upon a number of other centres, or of two sets of centres upon each other; in other words, the idea of mass (which, as we have seen, Art. 60, means primarily the number of centres) comes here into the problem.

Thus, let us take any mass, M, and suppose it to be acted upon solely by the accelerating force of the earth's attraction—in other words, to be let fall towards the earth. By Art. 67, the value of this accelerating force is 32·2 feet per second, a value which is always represented by g. But moving force is represented by the product of the mass and the velocity generated in a second; hence the moving force of gravity on the mass M is represented by Mg. Now the figure representing the moving force must be proportional to that representing the statical force, since they are the same thing, except that in the one case the cause produces motion, and in the other it only tends to produce motion (for which reason they are measured in different units). Hence, if W be the statical force acting on the mass—in other words, its weight in pounds—then by the meaning of proportion, we must have

$$W = C M g,$$

where C is a constant.

It is convenient to get rid of this constant by assuming such a value for the unit of mass as shall make the constant unity; so that the equation may be $W = Mg$. To fix what this unit of mass must be, let us take $W = 1$ lb. Then, the equation will be $1 = Mg$, and in order to make the equation become $1 = 1$, we must have $M = \frac{1}{g}$. But M is the mass of 1 lb. Therefore, the mass of 1 lb. is represented by $\frac{1}{g}$, or is $\frac{1}{g}$ of the unit of mass; hence the unit of mass

is the mass of 32·2 lbs., instead of the mass of 1 lb., as we might otherwise have assumed it.

The absolute unit of mass would naturally be the number of centres of force in the body which is the unit of weight: but this cannot be determined. The unit here given—viz., the mass of g lbs.—is the most convenient. It enables us (from the general formula $P = Mf$) to lay down the following law:—

If P be any force measured statically, and M the mass it acts upon, the formula for accelerating force may be applied by simply taking $\frac{P}{M}$ as the value of the acceleration.

126. Secondly, we have no longer merely to do with two centres of force, and two forces acting in opposite directions and equal to each other. We now include the case where there is a third force, or set of forces, which acts upon one of the first two, but does not act on the second. To make this clear, let there be two centres or particles, A and B. Then in the case of accelerating forces, Art. 84, we considered only the mutual actions of A and B, and the motion of B with regard to A, as fixed. But we may now suppose that there is a third point, C, which exercises an action upon B, and thus modifies its motion with regard to A, but which exercises no action upon A. This latter fact may be supposed to arise in several ways. Thus C may be supposed to be much farther from A than from B, and the forces to diminish so rapidly with the distance, as to make the force between C and A insignificant as compared with that between C and B. For example, if we suppose one end of a bar of iron, 6 inches long, to be brought close to a magnet, the magnetic attraction on the farther end would be insensible as compared with that on the nearer. Or again, we may suppose A to be absolutely fixed in space by some other force, which does not affect B or C. For example, in terrestrial mechanics the centre of the earth is regarded as being thus fixed, no motions of any ordinary bodies being able to produce any appreciable effect upon it.

127. Taking the simplest case first, we will suppose that there are only two bodies, A and B, to be considered, and that the mass

of the moving body, B, is represented by *m*. The action of A then takes the form of a certain unbalanced force, F, acting upon the mass *m*. We will assume that this force is constant. This assumption is not, of course, absolutely true for any of the known forces of nature, if they act for a finite time; but is practically true for many of them,—*e.g.*, the attraction of the earth on bodies at its surface,—and is practically true for all, if the interval of time considered is exceedingly small.

128. Making these assumptions, the fundamental question to be answered in this department is the following:—*A constant unbalanced force, F, acts on a particle of mass* m, *which is in motion during the action; how are we to measure the total effect of the force upon the particle when the action has lasted for a given time?*

129. The answer to this question may be stated at once. It is that the effect is measured, with regard to the force, by *The product of the force and of the distance moved through by the centre or body, in the direction of the force, during the action.* And this product is called the Energy Exerted by the force. Again, with regard to the body moved, the effect is measured, if the body start from rest, by *The product obtained at the end of the action, by multiplying together half the mass and the square of the velocity in the direction of the force;* or by the difference between this product and the similar product taken at the beginning of the action, if the body has already a velocity in that direction when the action commences. This product is called the *vis viva*, or the kinetic energy, or the actual energy, or the energy of motion, of the body; and the effect of the force is therefore measured by the change in this quantity, whichever name be used for it, during the action.

130. These principles are laid down in many text-books without any explanation of the reason why the energy exerted—*i.e.*, the product of the force and the distance—is the proper measure of the effect of the force. Nor does it at first sight seem clear why the element of time should be altogether absent from the *measure of the effect*, or why this measure, referred to the body,

should be in terms of the square of the velocity, and not of the velocity simply.

131. To elucidate this, let us suppose that the force, instead of being continuous throughout the motion, acts discontinuously at certain equal small intervals of space ds; so that, at the beginning of each of these intervals, there is an instantaneous action which generates in the body exactly the same velocity as is really generated during that interval by the continuous force. Let the number of these intervals in any space s be n, so that $s = n\,ds$. Then the total effect on the body, while traversing the space s, will, by the second law of motion, be the sum of the n effects due to the action of the n equal impulses at the beginning of the n intervals. It is evident, therefore, that, so long as the strength of the impulses remains the same, the total effect will vary as n. But $s = n\,ds$, and ds is supposed always the same; therefore, the total effect will vary as s.

Again, if the number of the impulses remains the same, the effect will, of course, vary with the strength of each impulse; in other words, with the force.

Hence, by the ordinary principle of variation, if both the spaces and the forces be different, the effect will vary as their product. But by considering the length of each interval ds as indefinitely small, and therefore the number n as indefinitely great, we may make the assumed circumstances approach indefinitely near to those of a constant force acting continuously over the same space; and hence we may say that the effect of such a force will vary as the product of the force itself, however measured, and the distance through which it acts.

This, as already stated, is called the energy exerted by the force. It does not contain the element of time, because neither the number of impulses nor the strength of each impulse are in any way affected by the velocity with which the body passes over the successive intervals ds; in other words, by the time which it occupies in describing the whole distance s.

132. We have thus shown that if F be the moving force of a given centre A, and s the space through which it acts on any other body or *centre B, its* effect will be properly measured by the

product $\mathbf{F}s$. This assumes that F is constant. If F vary with the distance, as will always be true in nature, then the same holds for each indefinitely small element of space ds—*i.e.*, the effect of the force, while the moving centre B traverses this element, is measured by $\mathbf{F}\,ds$.

133. Let us now endeavour to express the same product in terms of the mass and velocity of the body B. Assume that the body starts from rest: let v be the velocity generated by the force in the time t, and let f be the velocity which would be generated by the force in a unit of time. Then f is a measure of the accelerating force acting on B, and by Art. 90 we have

$$v^2 = 2fs, \text{ or } fs = \frac{v^2}{2}.$$

Multiply both sides by the mass m; then we have

$$mfs = \frac{mv^2}{2}.$$

But mf represents the momentum generated in a unit of time, and, by Art. 65, this is the measure of the moving force F. Hence we may write

$$\mathbf{F}s = \frac{mv^2}{2}.$$

But by the last article $\mathbf{F}s$, or the energy exerted, represents the effect of the force F acting over the distance s. Hence we see that, as referred to the moving body B, this effect is measured by multiplying the mass into the square of the velocity, and dividing by 2. This quantity, $\frac{mv^2}{2}$, we call the kinetic energy of the body whose mass is m and velocity v.

134. As an illustration, let us suppose that two equal shots are fired successively out of the same gun, that the velocity of each is measured as it leaves the muzzle, and that the velocity of the second shot is found to be twice as great as that of the first. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the forces of explosion in the two cases, supposed to *be constant during* the motion of the shot up to the muzzle. Then,

since s and m are the same in both cases, the equations for the two cases will be

$$F_1 s = \frac{m v^2}{2},$$

$$F_2 s = \frac{m (2v)^2}{2}.$$

Hence we have $F_2 = 4 F_1$; in other words, the force of the explosion must have been in the second case four times what it was in the first.

135. It may appear at first sight that this is not consistent with the fact (Art. 29) that force is measured by the velocity generated. If this is so, and if the velocity generated is double in the second case, must not the force be double also, and not quadruple? But the student must remember that the measure of forces is velocity generated in the same *time*, not in the same *space*. Since space described is proportional to velocity, and since F_2 generates velocity twice as fast as F_1, it follows that the first shot will traverse the bore of the gun in half the time of the second. Hence, what F_2 has really done, in comparison with F_1, is to generate double the velocity in half the time; and to do this it must clearly be four times as great.

136. We have hitherto supposed that the point started from rest. Let us now suppose that it had, in the direction of the force, an initial velocity V. Then the equation of motion is (Art. 90)

$$v^2 = V^2 + 2fs,$$

or

$$fs = \frac{v^2 - V^2}{2}.$$

Multiplying by m as before, we have

$$F s = m \frac{v^2 - V^2}{2}.$$

But $\frac{m v^2}{2}$ is the *vis viva*, or kinetic energy, at the end of the time t, and $\frac{m V^2}{2}$ is the kinetic energy at the beginning of the same time.

Therefore the expression on the right hand represents the *change* which has taken place in the kinetic energy during the motion; and therefore the correct general statement of the fact here proved is as follows :—*If a constant unbalanced force act upon a moving body for a given time, its effect is measured, either by the energy exerted* (F*s*) *as regards the forces, or by the change in the kinetic energy* $\left(m\frac{v^2 - V^2}{2}\right)$ *as regards the body.*

[137. The proposition proved above may easily be generalised.

Problem.—*To find the effect of a force F, acting for an element of time,* d t, *upon a point P, whose co-ordinates are* x y z, *and which is moving with a velocity* v.

Suppose the co-ordinates of P, at the end of the time dt, to be $x + dx$, $y + dy$, $z + dz$; let vx, vy, vz represent the angles which P's direction of motion makes with the three axes respectively, and let Fx, Fy, Fz be the angles which F's direction makes with the same axes. Then, by the laws of the resolution of forces, Art. 121, we can resolve F into three forces—F cos Fx, F cos Fy, F cos Fz—parallel to the axes; and by the resolution of velocities, Art. 123, we can resolve v into three velocities—v cos vx, v cos vy, v cos vz—parallel to the same axes. Similarly, if dv be the velocity generated by the force, we can resolve $v + dv$ into $(v + dv)$ cos vx, $(v + dv)$ cos vy, and $(v + dv)$ cos vz respectively. Then, confining our attention to the axis of x, we have a point which, in the time dt, moves through a space dx under the action of a force F cos Fx, and with a velocity which increases from v cos vx to $(v + dv)$ cos vx. Hence, by Art. 136, the effect of the force will be measured, either by the energy exerted, F cos Fx dx, or by the change in the kinetic energy, $m\frac{(v + dv)^2 - v^2}{2}\cos^2 vx$.

There will be a similar expression for each of the other two axes, formed by simply changing x into y and z respectively.]

138. Hitherto we have assumed that there is no third force acting on the moving centre B so as to affect its motion towards the fixed centre A. This is what would be true if A and B were *the only centres in* the universe. In nature such a case, of course,

cannot occur. The number of centres in the mechanical universe is incalculable, and by the definition of matter these all act upon both A and B. There are cases, however, where the action of these extraneous centres, owing to their distance or other causes, is insignificant when compared with the direct action between A and B, and may for many purposes be neglected. One such case is that of a body falling to the earth *in vacuo*. Another is the case of impact, such as the blow of a hammer, where the action is almost instantaneous. In these cases the effects are represented with sufficient exactness by the expressions indicated above. But we must now go on to consider what modifications are introduced by the presence of other forces.

139. Let us take the simplest possible case, and assume that, in addition to the fixed centre of force, A, and the moving centre, B, there is a third centre, C, placed in the prolongation of the line A B, on the other side of B, and therefore acting upon B in the opposite direction to the action of A.* For further simplicity, we shall assume, (1.) that C and A have no mutual action; (2.) that B is initially at rest; (3.) that C as well as A is fixed; (4.) that the forces with which A and C act on B are constant, or only vary by amounts that may be neglected. We will consider hereafter how far these assumptions can be realised in the universe, and how far they affect the conclusions.

140. **Problem.**—*To find the net effect of two moving forces, acting in opposite directions upon the same particle.*

Let P and Q be the forces with which A and C respectively act upon B; and let P be greater than Q. Then, by the second law of motion (Art. 76), each of these forces will produce its full effect exactly as if the other was not present, and the net effect upon B will be simply the difference between these opposite effects. Since P is greater than Q, B will move towards A.

Let s be the distance through which B has moved towards A at the end of a given time. Then the effect of A will be measured, just as before, by the product $P s$. For, as in our former proof

* It is supposed throughout that the forces are attractive; if they are repulsive the demonstrations will not be affected, but B's motion will be in the opposite direction.

(Art. 131), we may imagine the action of A divided into impulses, which act at successive points in space separated by small intervals. These impulses are independent of the time occupied in describing the intervals between one point and another, because, by the definition of matter, the forces are not functions of time. Moreover, each of these impulses will produce its full effect, by the second law, independently of the action of C. Hence, as before, the total effect will vary jointly as the number and strength of the impulses, and will, therefore, be represented by Ps.

Let us now make a similar assumption with regard to the action of C, namely, that it is divided into impulses acting at the same points in space as those of A. Each of these impulses will produce its effect, but this effect will be neutralised, as regards B, by the opposite impulse due to A; and the net impulse actually given to B will be the difference between the impulses due to A and C respectively.

Let dv be the acceleration produced by one of the impulses given by A, and let n be the number of such impulses in a second. Now each impulse, being instantaneous, can be measured only by the total momentum it generates, that is, by $m\,dv$, where m is the mass of the body acted upon. And since the impulses are equivalent to the force, it follows that the sum of the momenta generated in one second by the impulses must equal the momentum generated in one second by the force. But this momentum is the measure of the force itself. Hence we have

$$n\,(m\,dv) = \mathrm{P}, \text{ or } m\,dv = \frac{\mathrm{P}}{n}.$$

By precisely the same reasoning we have

$$m\,dv_1 = \frac{\mathrm{Q}}{n}.$$

Here dv_1 is the acceleration due to each of the impulses given by C, and n is the same as before, because the impulses from A and C act simultaneously at the same place and at the same time. But *if we take* any pair of these impulses, they constitute two forces, *acting on the same* point in opposite directions, and therefore, by

Art. 97, the net effect is that due to the difference between them. But this difference is

$$m\,dv - m\,dv_1 = \frac{P - Q}{n}.$$

Hence the net effect of all the impulses measured during one second is

$$n\frac{P-Q}{n} = P - Q;$$

therefore the effect on the point B is the same as if it were acted upon continuously by a single force $(P - Q)$. Hence, by Art. 133, when the point P has described the space s, the net energy exerted will be represented, as regards the forces, by

$$(P - Q)s,$$

and as regards the moving point by

$$\frac{m\,v^2}{2},$$

where m is B's mass, and v its velocity at the end of the space s.

141. The various quantities considered in the foregoing fundamental investigation have all received names, with which it is necessary that the student should be acquainted. To some of them different names have been assigned by different writers: we shall give in the text those which have been generally adopted; but we shall mention the others in footnotes, in order that the student may not be at a loss, if he should hereafter consult books in which those terms are used.

142. (Def.) *If a point B is acted upon by a force P in one direction, and by a lesser force, Q, in the opposite direction* (Art. 140), *then the name usually given to P is the Effort, and that given to Q is the Resistance.*

Again, we have seen that the net effect is the same as if the point had been acted upon by a single force equal to the difference of the two acting forces, or $P - Q$. *This difference,* $P - Q$, *is called the Unbalanced Effort.*

Again, the gross effect of the effort P on B is represented by Ps,

s being the space moved through by B in the given time. This is called the *Energy Exerted by A upon B.

Again, the gross effect of the resistance Q on B is represented by $-Qs$ (the negative sign being required, because Q acts in the opposite direction to P). *This is called the †Potential Work done upon B.*

Again, the effect of the unbalanced effort $(P-Q)$ on B is represented by either of the expressions $(P-Q)s$, or $\frac{mv^2}{2}$. *This effect is called the ‡Kinetic Work done upon B.*

143. Now, with reference to these names, it may perhaps be asked by the student, why the potential work, $-Qs$, is not called the energy exerted by Q upon B, after the analogy of Ps? The answer to this is that this (as will be seen below, Art. 149) would not accord with the general definitions of work and of energy, upon which the above nomenclature is based. These definitions may now be given.

144. **Definition of Work.**—*If one centre of force, taken as fixed, acts upon a second centre, and if the second centre has a motion in the direction in which the force of the first tends to move it, then the first centre is said to do work upon the second; and the work done is measured by the product of the force and of the distance moved through in the direction of the force.*

145. **Definition of Energy.**—*Energy means the power of doing work; and the total amount of energy possessed by any centre of force is measured by the total amount of work which it is capable of doing upon all the other centres upon which it acts.*

146. It will be observed that work, as defined above, is measured in terms of the forces, that is, in terms of the causes producing, not of the effect produced. It may be asked, therefore,

* It has also been called the work accumulated or stored up in B, and also the total work done upon B by P.

† It has also been called the potential energy imparted to B, and also (by Rankine) the work done upon B simply.

‡ It has also been called the kinetic energy imparted to B, or the energy *of motion, or (by* Rankine) the actual energy imparted to B: and lastly, it is *known by the old term of vis viva.*

how are we to recognise and measure work done in cases where we do not know the force acting, but can only observe the effect on the body acted upon? The answer is that the effect is twofold. In the first place, there will in general be an opposing force, or resistance, to the point's motion in the direction of the moving force; and the motion of the body through a certain space against this resistance represents an effect, or work done, which is measured by the product of the resistance and the space described. This is, in fact, the commonest instance of what we call, in ordinary language, doing work; as when we lift a body against the force of gravity, push it along a plane against the resistance of friction, and so forth. But, secondly, we may also consider ourselves as doing work when we throw a stone, or shoot an arrow; in which cases the resistances to the initial motion are insignificant, but we expend our energies (as we should say) in the simple fact of giving motion. Thus, if in the general case considered, the point has had its velocity increased while passing through the given space, then work will also have been done upon it in producing that increase; and that work will be measured, as shown in Article 136, by the increase of the *vis viva*.

It is, of course, necessary that these two kinds of work, being quite distinct in character and measurement, should bear distinct names. Accordingly, the first is called the potential work, and the second the kinetic work,* as in Art. 142.

§ 12. Conservation of Energy.

147. Let us now consider the definition of energy. Since energy is the power of doing work, it follows that energy will be lost or expended whenever that power is diminished, gained whenever that power is increased. Now, as the body B approaches A, we suppose the action of A to be divided into a number of impulses, which act successively upon B. Each of these impulses

* In Rankine's text-books, *e.g.*, his *Applied Mechanics*, the term work is restricted to the first of these classes, *i.e.*, potential work, and must be so understood *wherever it occurs*.

has been given once for all, and has produced its proper effect in work, potential or kinetic, or both. It cannot be given again, for the continuance of A's action upon B will only cause B to come closer and closer to A, and can never bring it back to those particular distances at which those particular impulses are always ready to act. Hence A's power of doing work upon B has been diminished by the amount of the work done by that particular set of impulses: in other words, energy has been expended or lost.

148. But let us now consider what has happened with respect to the action of C upon B. The effect of the motion has been to cause a series of attractive impulses, due to C, to take place upon B, as in the case of the impulses due to A; but each of these impulses has been in the opposite direction to the motion, and it has been counteracted by the superior impulse due to A. At the end of the motion considered, B is farther from C than it was at the beginning, by the distance s.

Now, if we suppose the action of A annihilated, and the points B and C left to themselves, what will happen? The motion of B, being no longer kept up by the action of A, will be checked by the attraction of B, and finally stopped. Motion will then begin in the opposite direction—*i.e.*, towards C. In time B will come again to traverse the space, which we have already described it as traversing, but now in the reverse direction. Since the action of C, by our definition of matter, is independent of time, and is always the same at the same distance, it follows that the first impulse given to B, on its returning road, will be exactly the same in magnitude and direction as the last impulse which it received on its outward journey. Similarly the second impulse of the new set will be precisely the same as the last but one of the old set, and so on throughout. Thus, while traversing the space s, B will receive over again a series of impulses precisely the same in number and amount as those it received on the former journey; in other words, an amount of energy will be exerted by C upon B, represented, as before, by $-Q\,s$.

The power of expending this energy has been given to C by the *action of A, in drawing* B farther from C: so that, while A has

itself been *losing* energy, expended on B, it has *increased* C's energy (*i.e.*, by definition C's power of doing work upon B) by the amount $Q\,s$.

149. We now see why $-Q\,s$ cannot be termed energy expended by C upon B: viz., because, as a matter of fact, instead of an expenditure of energy by C upon B, there has been an increase of energy, due to the more powerful action of B, tending in the opposite direction.

150. We have here spoken of C as remaining fixed and B moving towards it, because that was the assumption with which we set out. But of course we may just as well assume B to be fixed, and to draw C towards it by the equal mutual attraction subsisting between them; or—which would really be the case if B and C were left to themselves—that they both move towards each other under the influence of that mutual attraction. The only difference will be, in the last case, that the distance s, instead of being measured along B's path only, will be measured partly along B's and partly along C's; being, in fact, in all cases, the distance by which the two centres have approached each other during the motion.

We may therefore say that, on our definition of matter, the potential work done upon B in the course of its motion renders possible the exertion of a precisely equivalent amount of energy, due to the mutual attractions between B and C.

151. We have thus proved that the energy lost by A in the case of the three points A, B, C, now considered, so far as it is expended in doing potential work, is balanced by an equivalent amount of energy gained by B; in other words, so far as potential work is concerned, there is no gain or loss of energy on the whole, if we take the three points A, B, C, as constituting one system. We go on to consider whether the same will hold of the kinetic work, $\frac{m}{2}v^2$.

152. To examine this question by itself, let us suppose that at the end of the space s the centres, A and C, are replaced by a single centre at C, acting with a force $(P-Q)$—that is, equal in amount, *but opposite* in direction, to the net force which has

generated the kinetic work, $\frac{mv^2}{2}$. As before, we may suppose this force, P – Q, to act by impulses at intervals ds. Then, by the second law of motion, each of these impulses will produce its full effect upon B, irrespective of the fact of B's present motion; it will therefore destroy a portion of B's kinetic energy precisely equivalent to that which was generated by any one of the n equal impulses which acted on B during its motion along the space s. Hence, by the time that n of these equal impulses, due to P – Q, have acted upon B, the whole of its kinetic energy will have disappeared, and it will be at rest. But in the meantime it has overcome the force P – Q through the distance $n\,ds = s$, precisely as the force Q was overcome through the distance s in the former motion; and therefore, as explained in Art. 148, the kinetic energy $\frac{mv^2}{2}$, in disappearing, must have generated an amount of potential energy, due to the mutual attraction between B and C, which is represented by $-(P - Q)\,s$. This is precisely equal in amount to the energy by which the velocity v was originally generated in B.

153. Here, as before, for the sake of clearness, we have represented the energy as being destroyed by precisely the same steps as those by which it was generated; but this representation is in no wise essential to the proof. We may suppose, for instance, that A and C are both annihilated, and that B flies on in a straight line, with undiminished velocity v, until it comes into the range of another centre, D, whose force, R, may be a repulsive one. The centre B will then be gradually stopped, and will come to rest in a distance S, which will depend on the value of R, but which will certainly be such that $RS = \frac{m}{2}v^2$; inasmuch as RS is known to represent the total effect which R will, during the passage over the space S, have had on B, and this effect has been that of destroying the whole of the kinetic energy, represented by $\frac{m}{2}v^2$.

We may, therefore, say that, on our definition of matter, the kinetic work done upon B in the course of its motion renders *possible the exertion* of a precisely equivalent amount of energy,

due to the mutual action between B and any other centre within whose range it may come.

154. Let us now gather our results together. We started with an amount, $P s$, of energy exerted by A. We saw that the effect of this exertion was the performance of work, but work under two different forms—namely, potential work, represented by $Q s$, and kinetic work, represented by $\frac{m}{2} v^2$. We then found that the performance of each of these amounts of work rendered possible the exertion of a fresh amount of energy, not due, like the first, to the action of A, but precisely equivalent in amount to the original energy exerted by A in the two cases. Hence, we see that, taking the system as a whole, there has been no gain or loss of energy during the action. This is the principle of the Conservation of Energy as applied to this particular case.

155. It will now be advisable to recur to the assumptions (Art. 139) with which we started on this investigation, and see how far they affect the generality of the principle we have just stated.

156. In the first place, we assumed that A is fixed. Since in every case of mechanics it is necessary to assume some fixed point, and to consider the motions relatively thereto, there is no difficulty in making A that point. In practice the centre of the earth may be considered as fixed for all questions of terrestrial mechanics, and the centre of the sun as fixed for the purposes of astronomy.

157. Secondly, we assumed that C is fixed. But if C have a motion in the direction C A, or the opposite, the only effect will be that we shall have to diminish or increase, as the case may be, the quantity $Q s$ (expressing the energy imparted by C) by the quantity $Q s_1$; where s_1 is the space described by C, during the time of the motion, either in the direction C A or the opposite. The effect will therefore be the same (taking the first case) as if C were at rest, while the amount of Q was diminished in the ratio $s - s_1 : s$. There would thus be a diminution in the potential work, and, of course, a corresponding increase in the kinetic work, done by A.

158. *Thirdly*, we supposed C and A to have no mutual action.

In reality this is not, of course, true, by our definition of matter; but in many cases C and A may be fixed with regard to each other —as where coals are wound up from a pit by a steam engine at the surface, which is fixed with regard to the earth—and this amounts to the same thing. Moreover, as we shall see hereafter, the forces of cohesion are very great at insensible distances, but are quite inappreciable at sensible distances; hence, in considering, for instance, the case of a rope in tension, we may treat any section as being influenced by the two sections on either side of it, but not by those beyond. If, however, A does act upon C, the effect is to move C in the direction of A, and thereby make the distance between B and C, at the end of the motion, less than in the former case. Thus, let R be the force which A acts on C, and S the distance through which C has moved along the line CA. Then A will have exerted the additional energy RS, which will all take the form of kinetic work done upon C.

On the other hand, the energy, $P s$, exerted on B will be just the same as before; but the part of it which takes the form of potential work will be reduced from $Q s$ to $Q(s - S)$, because the number of impulses distributed over the space S will not have been given by C.

Now the effect of this on B will be exactly the same as if, C remaining fixed, the strength of each impulse had been reduced in the proportion $s - S : s$; for in that case the total effect would be represented by $Q \frac{s - S}{s} \times s = Q(s - S)$ as before. But if the resistance be reduced from Q to $Q \frac{s - S}{s}$, then the unbalanced effort will be increased from $P - Q$ to $P - Q \frac{s - S}{s}$; and the kinetic work, due to this unbalanced effort, will be increased from $(P - Q) s$ to $\left(P - Q \frac{s - S}{s}\right) s$, or to $(P - Q) s + Q S$.

Thus the kinetic work will be increased by Q S, which is exactly the amount, as shown above, by which the potential work is *diminished*. Hence the assumption that A acts on C does not *introduce any gain* or loss of energy on the whole; the energy

exerted on C takes the form of kinetic work, and the energy exerted on B partly of kinetic and partly of potential work, as before, but divided in different proportions.

159. Fourthly, we assumed that B is initially at rest. Now, let us suppose that instead of being at rest, the point B has an initial velocity V in the direction B A. (If the velocity is in the opposite direction, B C, the demonstration will be the same, simply writing $-V$ for V throughout). Then, by virtue of this velocity, it will also have kinetic energy, represented by $\frac{m}{2}V^2$, which can be converted into potential work, as explained in Art. 152. Let t be the interval of time considered. Then, by Art. 91, since the net moving force $(P-Q)$ has been acting on the mass m during the time t, it will have generated in B—irrespective of B's initial motion—a velocity represented by $\frac{P-Q}{m}t$, and will have caused B to describe a space represented by $\frac{1}{2}\frac{P-Q}{m}t^2$. In addition to this B will have described, by virtue of its initial velocity, a space Vt. Hence the total energy exerted by A will now be represented by $P\left(Vt+\frac{1}{2}\frac{P-Q}{m}t^2\right)$; and the total amount of energy which has to be accounted for at the end of the motion, is

$$\frac{m}{2}V^2+P\left(Vt+\frac{1}{2}\frac{P-Q}{m}t^2\right).$$

Now the energy left at the end of the motion is as follows:—
(1.) The potential energy, due to the potential work done in moving B through the space $\left(Vt+\frac{1}{2}\frac{P-Q}{m}t^2\right)$ in opposition to the force Q. This is represented by

$$\left(Vt+\frac{1}{2}\frac{P-Q}{m}t^2\right)Q.$$

(2.) The kinetic energy, due to the final velocity of B, or to $\left(V+\frac{P-Q}{m}t\right)$. This is represented by

$$\frac{m}{2}\left(V+\frac{P-Q}{m}t\right)^2=\frac{m}{2}V^2+VPt-VQt+\frac{(P-Q)t^2}{2m}P$$
$$-\frac{(P-Q)t^2}{2m}Q.$$

Adding the two expressions, we get for the energy left

$$\frac{m}{2}V^2+VPt+\frac{(P-Q)t^2}{2m}P.$$

This is exactly the same expression as that given above for the total energy which has to be accounted for. It appears, therefore, that there is again no loss of energy during the motion, and therefore the principle of the conservation of energy is not affected by the initial velocity of B.

160. Lastly, we assumed that the forces P and Q are constant. This of course is never exactly true in the universe, although it is true within our limits of measurement in many cases—*e.g.*, that of a stone falling to the earth. But we may always assume it to be true for an indefinitely small interval of time. Hence, for each such interval the conservation of energy will hold, and if so, it must also hold for the sum of the intervals; that is, for any particular time that is considered. The energy exerted must, of course, be determined in this case by the methods of the integral calculus.

161. We have thus proved that the principle of the conservation of energy is true, with complete generality, for the case in which there are only three centres of force, A, B, C, lying in the same straight line.

So far as we have hitherto proved the principle, it may be expressed by saying :—"If we consider a system composed of three isolated centres of force, in the same straight line, then whenever energy, or the power of doing work, is lost by one of those centres, in consequence of their motions and mutual actions, an equivalent amount of energy will be added to one of the others, or to both combined; and thus there is no loss or gain of energy in the system as a whole.

[From this we may deduce the truth of the general principle *of the conservation* of energy, as follows.

162. **Theorem.**—*In any system of matter in motion under its own forces, the total energy, potential and kinetic, in the system remains constant.*

We have already proved this for the case of a point acted on by two other points, one on each side of it. We now proceed to show that the case of a particle acted on by any number of other particles, in any position whatever, may be reduced to the above case; and that the proposition is therefore generally true.

Take the position of the particle, O, at the instant under consideration, as the origin of co-ordinates, and let x_1, y_1, z_1, be the co-ordinates of any other particle, P, whose distance is r_1, and which acts upon it. Then by the definition of matter, the mutual force between P and O varies only with the distance, and may therefore be expressed by the symbol

$$m\, m_1 f(r_1),$$

where $m\, m_1$ are the masses of O and P respectively, and $f(r_1)$ expresses any given function of r_1.

Now, to resolve this force parallel to the three axes, we have only to multiply this expression successively by the cosines of the angles which r_1 makes with the axes, that is, by $\frac{x_1}{r_1}$, $\frac{y_1}{r_1}$, $\frac{z_1}{r_1}$. Hence the resolved part along the axis of x is

$$m\, m_1 \frac{f(r_1)}{r_1}\, x_1.$$

Let there be any number of similar points, whose co-ordinates are $x_2\ y_2\ z_2$, $x_3\ y_3\ z_3$, &c., and let them be resolved in a similar manner. Then the total force acting on O along the axis of x in the *positive* direction (assuming x_1, x_2, &c., to be all positive) is

$$m\left[m_1 \frac{f(r_1)}{r_1} x_1 + m_2 \frac{f(r_2)}{r_2} x_2 + \ldots\right]$$

Now, let us suppose that there is a point distant X_e from the origin, and such that

$$(m_1 + m_2 + \ldots) \frac{f(X_e)}{X_e} X_e = m_1 \frac{f(r_1)}{r_1} x_1 + m_2 \frac{f(r_2)}{r_2} x_2 + \ldots$$

Then it is evident that the action on O, in the positive direction of the axis of x, is the same as if it were acted on by a single particle of mass $(m_1 + m_2 + \ldots .)$ situated on the axis of x, at a distance X_s and having the same law of force, viz., $f(X_s)$.

By precisely similar reasoning, the action on O in the negative direction of the axis of x, by particles M_1, $M_2 \ldots$, may be reduced to a single force, $m(M_1 + M_2 + \ldots) f(X_r)$, acting at a distance X_r from the origin, in the negative direction.

But the case is thus reduced to a particle acted on by two particles in the same straight line, one of the forces forming the effort, and the other the resistance, and both varying according to the same law; and for such a case the conservation of energy has been proved to hold.

Therefore the principle holds generally for the resolved part of the forces along the axis of x; and similarly it may be shown to hold for the resolved part of the forces along the axis of y and z. And by the second law of motion the actions of these resolved parts, considered independently, make up together the total action of the forces; hence the principle holds generally.]

PART II.—STATICS.

§ 1. Conditions of Equilibrium for Forces acting on One Point.

163. In the first part of this treatise (Art. 120), we solved the general problem of the resultant of any number of forces acting together upon one point. Let us suppose this resultant, R, to be found in any particular case, and let us now apply to the point another force, – R, equal in magnitude and opposite in direction to R. Let us also assume that the point is initially at rest. Then, by the principle of symmetry (Art. 99), the point will remain at rest, because there is no reason why it should move, under the action of the forces, in the direction R, rather than in the direction – R. The resultant of the whole system of forces is then R – R, or zero.

164. Now when a number of forces, acting on a body, keep it at rest, and therefore have a resultant = zero, they are said to be in equilibrium * (Art. 95); and the position of a body, which is such that the forces acting upon it are in equilibrium, is said to be a position of equilibrium.

165. The science which treats of points or bodies when in the condition of equilibrium—in other words, when the forces acting upon them have no resultant—is called the science of Statics.

166. We proceed to consider the condition of equilibrium for a point acted on by forces in various circumstances. For the present, we assume that all the forces are in the same plane.

167. If only one force acts on the point, it is obvious that there cannot be equilibrium.

* The word equilibrium (being derived from the Latin words for equal and scale) refers to the common case of a balance, which is in equilibrium *when the weights in the two* scales are equal.

168. **Theorem.**—*If two forces act on a point in equilibrium, they must be equal and opposite to each other.* For, first, let them be opposite, but not equal. Then, by Art. 140, the point will move in the direction of the greater force, and as if it were acted upon by the difference between the two forces. Secondly, let them be equal, but not opposite. Then, by the parallelogram of forces (Art. 106), they will have a resultant, which is represented by the diagonal of any parallelogram of which they represent the sides; therefore the resultant is not zero, and therefore the forces are not in equilibrium.

169. **Theorem.**—*If three forces act on a point, they will be in equilibrium if they are represented in magnitude and direction by the sides of a triangle taken in order.*

Let A B D be a triangle, whose sides, A B, B D, D A, taken in order, represent in magnitude and direction the forces A B, A C, D A, acting at the point A. Complete the parallelogram A B C D. Then, by Art. 106, the forces A B, A C, have a resultant A D: that is, the forces A B, A C, D A, are equivalent to A D, D A, and, therefore, balance each other.

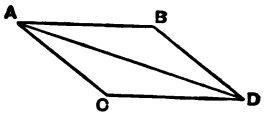

Fig. 10.

It follows that the forces represented by A B, B D, D A, would be in equilibrium if they were applied directly at the point A.

The converse is also true—viz., that if the forces balance, and the triangle be constructed, its sides will be proportional to the magnitudes of the forces.

170. The above proposition is usually quoted as the Triangle of Forces. It supplies a very convenient method of estimating the forces in practice, for any case of equilibrium where three forces alone are acting. If we know the directions of the forces, and the amount of one of them, we have only to draw a triangle, whose sides are parallel to the directions, and make the side parallel to the known force represent it to any convenient scale. Then the other sides will represent, to the same scale, the forces *parallel* to them respectively.

171. Theorem.—*If any number of forces* act on a point, they will

be in equilibrium if they are represented in magnitude and direction by the sides of a polygon taken in order.

In the polygon ABCDE, we know by Art. 169, that AB, BC are equivalent to a force represented by AC: similarly AC, CD are equivalent to AD, and AD, DE are equivalent to AE.

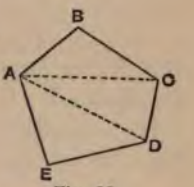

Fig. 11.

Therefore AB, BC, CD, DE, EA, are equivalent to AE, EA, and will balance each other. And the same will hold for any larger number of sides. Hence the proposition is true.

172. It follows from the triangle and polygon of forces, that any conclusions, established by geometry concerning the relations of the sides and angles of a triangle or a polygon, may be extended to the magnitudes and directions of forces in equilibrium. Thus we may conclude, from *Euclid*, i. 20, that if three forces be in equilibrium, any two must together be greater than the third. Again, if the directions of two of the forces are at right angles, we conclude, by *Euclid*, i. 47, that the square of the third force is equal to the sum of the squares of the other two. Again, by trigonometry, the sides of a triangle are respectively proportional to the sines of the opposite angles. Hence, if P, Q, R be three forces in equilibrium, and (QR) represent the angle between Q and R, &c., we have

$$\frac{P}{\sin (QR)} = \frac{Q}{\sin (PR)} = \frac{R}{\sin (PQ)}.$$

173. The polygon of forces gives a geometrical condition for the equilibrium of any number of forces acting on a point, but it is not an analytical one, nor always easy of application. This defect is supplied by the following proposition.

Problem.—*To find the conditions of equilibrium of any number of forces acting on one point and lying in one plane.*

By Art. 119 we know that the resultant of any number of forces acting on one point and lying in one plane, is given by

$$R^2 = X^2 + Y^2,$$

where X and Y are the algebraical sums of the components of the

several forces, taken along any two rectangular axes. Now, if there is equilibrium, we must have

$$R = 0.$$

But by the form of the equation this is impossible, unless both $X = 0$ and $Y = 0$.

[174. Similarly, if the forces are not all in one plane, we must have, by Art. 121,

$$X = 0,\ Y = 0,\ Z = 0.$$

Hence, the general condition of equilibrium is that the algebraical sums of the components of the forces, along any three rectangular axes, must separately vanish.]

§ 2. Conditions of Equilibrium for a Rigid Body.

175. Hitherto we have only considered the action of forces all acting on the same point. We now go on to consider the case of forces acting on several points, which are so connected together as to form a single body. This introduces several new conceptions, which require to be defined.

176. **Definition.**—*A rigid body is a collection of material points or particles, the relative positions and distances of which, with regard to each other, are supposed to be absolutely fixed, so that no external force can alter them.*

177. There is no such thing in nature as a body which is even approximately rigid. A steel bar fulfils the condition as well as almost anything; but even a steel bar is extended or bent, under the influence of very moderate external forces, through distances which can be easily observed, and even measured by the aid of delicate instruments. At the same time such *strains*, or alterations in the internal arrangement of a body, are often numerically exceedingly small, as compared with the forces applied, or with the changes of position in the body as a whole, relatively to other things; and in such cases they may often be neglected, and the body treated as if it were really rigid. By this means conclusions

may be drawn, which are correct for all practical purposes; whereas, if the strains in the interior of the body were taken into account, the problems would be rendered so much more complicated and difficult as to be unintelligible to the ordinary student, and sometimes quite insoluble. It must be remembered by the student that, while the methods of physics are generally exact, *there is practically no such thing as exactness in nature;* there are always small variations and sources of error, which must at some time or other be carefully investigated, although for many purposes they may be neglected altogether.

178. **Definition.**—*When all the points making up a rigid body are in motion in the same direction, and with the same velocity, the motion is said to be one of translation.*

179. **Definition.**—*When, of the points making up a rigid body, those which lie in one straight line are at rest, and the others move in circles round that line as an axis, then the motion is said to be one of rotation.*

180. If an axis in a body be fixed, then a motion of rotation is the only motion possible. For if any point in the body moves parallel to the fixed axis, or moves at right angles to that axis in any curve but a circle, it alters its distance from the particles composing that axis; which by the definition of rigidity it cannot do.

181. It is, of course, possible for a body to have a motion of translation and rotation at the same time. A familiar instance is a rifle-bullet, which moves through the air as a whole, while it is at the same time twisting rapidly about its longitudinal axis. Another case is a screw-propeller, which turns about its axis at the same time that it partakes of the forward motion of the ship. As a matter of fact any motion whatever of a rigid body, at any particular instant, may be treated as made up of a motion of rotation and translation combined. In such cases the axis of rotation is not fixed as regards space, but it is fixed as regards the direction in which the motion of translation takes place, being parallel to that direction; and the points in that axis have that motion of translation only.

If the body be treated as being in one plane, or in two dimen-

sions only, the axis of rotation becomes a point in that plane, and the other points in the body describe circles round that point.

182. In the case of a single point, such as hitherto we have considered, it is sufficient for equilibrium that there be no motion of translation; for the motion of one point round another cannot exist where there is only one point existing. But in the case of a body or system of several points, it is necessary for equilibrium that there should neither be a motion of translation nor of rotation.

183. Problem.—*To find the conditions of equilibrium for a rigid body or system consisting of two points only.*

In the case of two points the hypothesis of rigidity merely means that the distance between the two points is unalterable. Let A, B (Fig. 12), be the two points, and resolve the external forces acting upon them in directions along and perpendicular to A B. Let the resultant of the components perpendicular to A B be P and Q, and let the resultant of those along A B be R and S. Then, since A tends to move, under the influence of P, in the direction at right angles to A B, and therefore without either increasing or diminishing the distance A B, it follows that the fact of rigidity will not prevent its motion. Therefore if P has any real value, A will move. Hence the first condition of equilibrium is

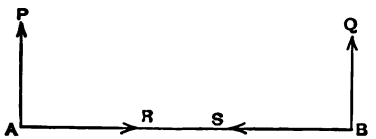

Fig. 12.

$$P = 0.$$

By similar reasoning, it follows that we must also have

$$Q = 0.$$

Next, turning to the forces which act along A B, it is evident by the principle of symmetry that there will be equilibrium if R = S, since there is then no reason why the system should move in one direction rather than the other. Hence the condition of equilibrium is

$$R = S.$$

These conditions may be summed up in saying that there must be no forces acting at right angles to A B, and that the forces acting parallel to A B must be equal and opposite to each other.

184. In the above proof we have assumed that both A and B are free to move. We may, however, suppose that one of them, as B, is absolutely fixed in space. Then the force acting on A, in the direction A B, may be any whatever; for A can only move along A B by altering the distance A B, and by the definition of rigidity this is impossible. The only condition for equilibrium in this case is therefore

$$P = 0.$$

185. **Theorem.**—*In any rigid body, if there be no motion of rotation about a certain point, the resultant of the forces acting upon the body must pass through that point.*

For if not, draw a perpendicular from the point upon the direction of the resultant. Then we may consider the system to be reduced to a system of two points, namely, the fixed point and the foot of the perpendicular drawn upon the resultant; and the force acting upon the latter will have a component at right angles to the line joining the two points. But, by Art. 183, this is impossible if there is to be equilibrium. Hence, if there is equilibrium, the resultant of the forces must pass through the fixed point.

186. We shall now consider the conditions of equilibrium for a body or system consisting of three points; and we will first assume that one of these points is fixed, and find the condition that the other two may not rotate round it. Practically, this is the same problem as that of a rigid rod, which can turn about one point in its length, and is acted on by forces at two other points, the weight of the rod being neglected. Owing to the great importance of this particular case, we shall present the problem under this form, as is usual in works on Statics.

187. **Definition.**—*A rigid rod, movable about a fixed point in its length, is called a lever. The fixed point is called the fulcrum, and the parts between the fulcrum and the extremities the arms.*

188. **Definition.**—*The moment of a force with respect to a given*

point, is the product of the force and the perpendicular from the point on its direction.

189. **Theorem.**—*If two forces, acting at the extremities of a lever and tending to twist the lever opposite ways, produce equilibrium, the moments of the forces about the fulcrum are equal.*

Let P, Q be the two forces (Fig. 13) acting at A and B, the extremities of a lever. Produce the directions of P and Q until they meet in C: take Cm, Cn proportional to P and Q, and complete the parallelogram $Cmpn$; join Cp and produce it to cut AB in O. Then the resultant of P and Q acts in the direction CO, and therefore O must be the fulcrum (Art. 185), otherwise there could not be equilibrium. Draw OD, OE perpendicular to AC, BC; then

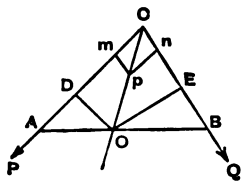

Fig. 13.

$$\frac{P}{Q}=\frac{\sin Cpm}{\sin mCp}=\frac{CO\sin BCO}{CO\sin ACO}=\frac{OE}{OD};$$

$$\text{Therefore } P\times OD=Q\times OE,$$

or the moments of P and Q about O are equal.

190. If the forces act parallel to each other, their directions will not meet, as they are supposed to do in the preceding proposition; in this case we must proceed as follows. We shall suppose, though it is not necessary, that the forces act perpendicular to the lever.

At A and B (Fig. 14) apply any two equal and opposite forces S in the direction of the lever; this, by symmetry, will not affect the equilibrium. Then the resultant of P and S will be some force in the direction CA suppose, and that of Q and S some force in the direction CB. Suppose them both to act at C, and there to be resolved into their constituent parts P and S, Q and S; the portions S, S will counteract each other, leaving a resultant P + Q in the direction

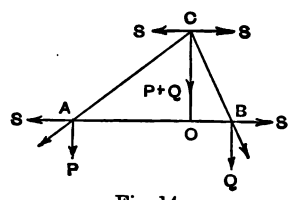

Fig. 14.

C O parallel to the directions of the forces. As before, O must be the fulcrum.

Then the sides of the triangle A O C are parallel to the directions of the forces P, S and their resultant; hence

$$\frac{P}{S}=\frac{CO}{AO},$$

in like manner,

$$\frac{Q}{S}=\frac{CO}{BO};$$

$$\text{Therefore } P \times AO = Q \times BO.$$

The same method is applicable, when the forces are not perpendicular to the arm.

If we had first proved the proposition now under consideration for the case of parallel forces perpendicular to the arm, it would have been very easy to deduce the more general case. For if in Fig. 13 we suppose the forces P and Q to be each resolved into two, one parallel and the other perpendicular to the arm, it is manifest that in the latter components of P and Q only is there any tendency to twist the lever. Now these latter portions are respectively P sin C A O, Q sin C B O; hence

$$P \sin CAO \times AO = Q \sin CBO \times BO,$$

$$\text{or } P \times OD = Q \times OE,$$

as was proved before.

191. In the above article it is clear that the fact of the three points, A, B, O, lying in the same straight line, is in no way essential to the proof. Thus we may suppose the bent line D O E (Fig. 13) to be the form of the lever, without in the slightest degree altering the demonstration. Hence the principle is applicable to a system of any form consisting of three points only.

192. **Problem.**—*To find the conditions of equilibrium for a system consisting of three points only.*

Taking the same diagram (Fig. 13), let D O E be the three points, and let the two resultants of the forces acting on D and E respectively meet, as before, in C. Then C*p* represents this resultant in

magnitude and direction; and, as before, if there is equilibrium, the direction of Cp must pass through O. Let the other forces acting upon O have a resultant, R. Then O is in the condition of a single point, acted on by two forces, Cp and R; and, by Art. 168, the condition of equilibrium is that these forces are equal and opposite. If these two conditions hold, O will have no motion of translation, that is, it will be fixed; and D and E will have no motion of rotation round it. Hence, the whole system will be at rest—in other words, there will be equilibrium. Hence the following are the conditions of equilibrium: *The resultant of the forces acting upon any two of the points must pass through the third, and must be equal and opposite to the resultant of the forces acting upon that third.*

If the resultants of the forces on D and E respectively are parallel to each other, the demonstration will be the same, using Fig. 14 instead of Fig. 13.

193. **Theorem.**—*The proper measure of the power of a force, as tending to produce rotation of a body about a fixed point, is the moment of the force about that point.*

Taking Fig. 13, as before, we have the general equation

$$P \times O D = Q \times O E.$$

Suppose P and O D to vary, but O E to remain constant, and to be a unit of length. Then the turning power of P will always be represented by that value of Q which makes the equation hold; since, on giving Q this value, the body will be in equilibrium, and therefore the tendency of P to turn the body round O will be counteracted.

First, suppose O D to remain the same, and alter P to m P, where m is any quantity whatever. Then, if Q_1 be the new value of Q, we have

$$Q_1 = m P \times O D = m Q.$$

Hence, when P becomes m P, the turning effect becomes m Q; that is, when the arm O D remains constant, the turning effect varies as the force P.

Similarly, if P remains the same, and O D is increased to $m \times$ O D, we have

$$Q_1 = m\,Q,$$

or, when the force remains constant, the turning effect varies as the arm.

Hence, by the ordinary law of variation, the turning effect varies generally as the product of the force and the arm—*i.e.*, as the moment.

194. On account of the property just proved, the conception of moments is one of very special importance in mechanics, and the student is recommended to take special pains to make himself completely master of it. The following extension of the principle of equality of moments is very important.

195. **Principle of moments stated generally.**—What has been inferred as to P and Q will be true however we increase the number of the forces. In order to arrive at the complete turning effect of a number of forces acting in one plane, it suffices to add their separate moments, for in doing so we are only adding numbers of the same kind. Moments tending to turn the body in one direction must be taken as positive, and in the opposite direction as negative.

The principle of the lever may therefore be stated generally as follows.

If any number of forces acting on a rigid body in one plane tend to turn it about a fixed axis, there will be equilibrium when the sum of the moments of the forces acting to turn the body in one direction is equal to the sum of the moments of the forces acting to turn it in the opposite direction. In other words, the algebraical sum of all the moments must be zero.

That point in the axis about which the moments are estimated is often called the *centre of moments*.

196. **Problem.**—*To find the resultant of two parallel forces.*

In Art. 190 we have shown that the resultant of two parallel forces, P and Q, acting in the same direction, is a single force equal to their sum, or P + Q, and cutting the line A B perpen-

dicular to the forces, in a point O, such that $P \times AO = Q \times BO$, *i.e.*, such that the moments about O are equal.

Let us now suppose that the forces P and Q act in the opposite direction (Fig. 15). Apply two equal and opposite forces, S S, as

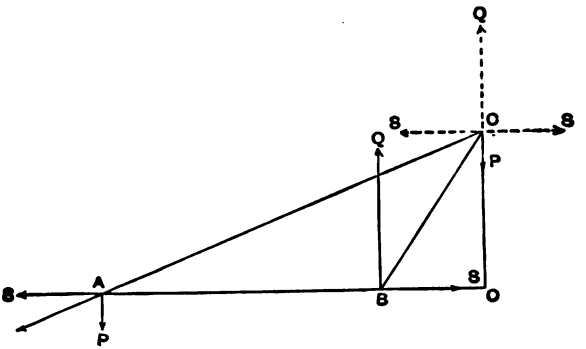

Fig. 15.

before, and make the same construction. Then the resultant of P and Q will evidently be a force (Q – P) acting in the direction O C. Also, we have, as before,

$$\frac{P}{S} = \frac{CO}{AO}, \text{ and } \frac{Q}{S} = \frac{CO}{BO};$$

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Suppose P and O D to vary, but O E to remain constant, and to be a unit of length. Then the turning power of P will always be represented by that value of Q which makes the equation hold; since, on giving Q this value, the body will be in equilibrium, and therefore the tendency of P to turn the body round O will be counteracted.

First, suppose O D to remain the same, and alter P to m P, where m is any quantity whatever. Then, if Q_1 be the new value of Q, we have

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Hence, when P becomes m P, the turning effect becomes m Q; that is, when the arm O D remains constant, the turning effect varies as the force P.

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The principle of the lever may therefore be stated generally as follows.

If any number of forces acting on a rigid body in one plane tend to turn it about a fixed axis, there will be equilibrium when the sum of the moments of the forces acting to turn the body in one direction is equal to the sum of the moments of the forces acting to turn it in the opposite direction. In other words, the algebraical sum of all the moments must be zero.

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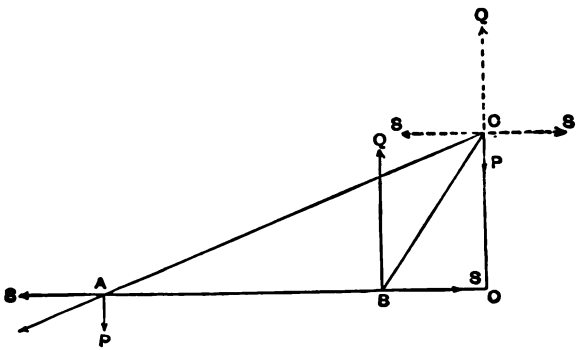

Fig. 15.

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acting in opposite directions, but not in the same line, is represented by an infinitely small force at an infinitely great distance; and therefore it has no finite value. There is no single force which will balance such a pair of forces, or prevent them from having their effect in causing rotation in the body to which they are applied. This effect requires to be further developed.

198. Definition.—*A pair of equal and parallel forces acting in opposite directions, is called a couple. The perpendicular distance between the forces is called the arm; and the product of either force into the arm is its moment.*

199. Definition.—*A straight line drawn perpendicular to the plane of a couple, and proportional in length to its moment, is called the axis of the couple.*

200. Theorem.—*The effect of a couple is completely represented by its axis.*

1. The plane in which the forces act is perpendicular to the axis, and therefore the direction of the axis determines the plane of the couple.

2. Let the forces P, P, act at the ends of the arm A B, and let the axis cut the plane of the couple, either in A B, or A B produced, as at E, F.

The turning effect round E

$$= P \times A E + P \times E B = P \times A B.$$

So also, the turning effect round F

$$= P \times F B - P \times F A = P \times A B;$$

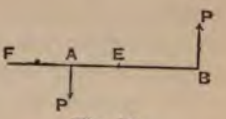

Fig. 16.

and this is equally true in the extreme case when E coincides with either A or B.

Hence the turning effect of the couple is proportional to the moment P × A B, which is proportional to the axis.

201. Since a couple is correctly represented in all respects by its axis, which is a straight line fixed in magnitude and in direction, it follows that we may apply to couples the same principles of composition and resolution which we have proved to be true in the case of simple forces. Also we can add and subtract the *parallel axes of a set of* couples just as we add or subtract simple

forces acting in the same straight line. By this process we obtain what is called a *resultant couple.*

202. We are now in a position to attack the general problem, which we may take as the fundamental one of statics,—viz., the conditions of equilibrium of any number of forces, acting on a rigid body or system of any number of points. Without the conception of moments and couples this cannot be solved.

203. **Problem.**—*To find the conditions of equilibrium of any number of forces acting in one plane upon different points of a rigid body.*

Let A be any point in the body, $P_1 P_2$, . . . the forces acting on the body, $\alpha_1 \alpha_2$ . . . the angles their directions make with a fixed line, $X A X_1$.

Let P_1 act at B, and apply at A two opposing forces, each equal and parallel to the force P_1; by symmetry, this will not disturb the equilibrium. Draw $A D_1$ perpendicular to $B P_1$. Then P_1 at B is represented by P_1 at A, parallel to $B P_1$, and by the couple $P_1 P_1$, whose moment is $P_1 \times A D_1$.

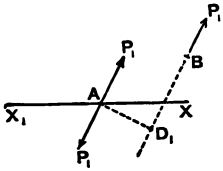

Fig. 17.

The remaining forces, P_2, P_3 . . . may be treated in like manner. We thus obtain a set of forces, P_1, P_2, P_3 . . . acting at A in directions parallel to their actual directions, and also a set of couples whose axes are parallel.

The couples are represented by their moments $P_1 \times A D_1$, $P_2 \times A D_2$, &c., and are equivalent to a single resultant couple whose moment is

$$P_1 \times A D_1 + P_2 \times A D_2 + \&c.$$

The forces acting at A must be in equilibrium. Hence, by Art. 119,

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \&c. = 0 \quad . \quad . \quad . \quad (1.)$$

$$P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \&c. = 0 \quad . \quad . \quad . \quad (2.)$$

Also, the resultant couple must disappear; therefore,

$$P_1 \times A D_1 + P_2 \times A D_2 + \&c. = 0 \quad . \quad . \quad . \quad (3.)$$

These three conditions are necessary and sufficient for the equilibrium of a rigid body under the action of forces in one plane. Since the line $X A X_1$ is any whatever, they are expressed by saying that *the sum of the components of the forces in any direction must be zero, and the sum of the moments of the forces round any point must be zero.*

204. **Corollary 1.**—In the foregoing proof we have in effect stated that a force P acting at the point D, in the arm A D of a lever, and tending to turn it about A, produces a push at A, which is equal to P.

Corollary 2.—If any number of forces acting on a rigid body in one plane tend to turn it about a fixed axis, there will be equilibrium when the algebraical sum of all the moments about that axis is zero.

[205. **Problem.**—*To find the resultants of any number of forces acting in any directions on a rigid body.*

Let the forces be referred to three rectangular axes Ox, Oy, Oz

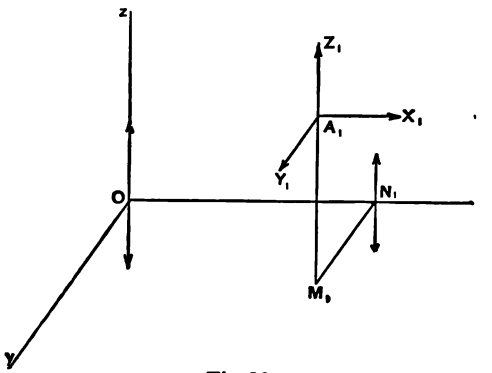

Fig. 18.

(Fig. 18); and suppose P_1, P_2, P_3,... the forces; let x_1, y_1, z_1 be the co-ordinates of the point of application of P_1; let x_2, y_2, z_2 be the co-ordinates of the point of application of P_2; and so on.

Let A_1 be the point of application of P_1; resolve P into components X_1, Y_1, Z_1, parallel to the co-ordinate axes. Let the

direction of Z_1 meet the plane of (x, y) in M_1, and draw M_1N_1 perpendicular to Ox. Apply at N_1 and also at O two forces each equal and parallel to Z_1, and in opposite directions. Hence Z_1 at A_1 or M_1 is equivalent to Z_1 at O, and two couples, of which the former has its moment $= Z_1 \times N_1M_1$, and may be supposed to act in the plane of (y, z), and the latter has its moment $= Z_1 \times O N_1$ and acts in the plane of (z, x).

We shall consider those couples as positive which tend to turn the body round the axis of x from y to z, also those which tend to turn the body round tho axis of y from z to x, and those which tend to turn the body round the axis of z from x to y.

Hence Z_1 is replaced by Z_1 at O, a couple $Z_1 y_1$ in the plane of (y, z), and a couple $-Z_1 x_1$ in the plane of (z, x). Similarly X_1 may be replaced by X_1 at O, a couple $X_1 z_1$ in the plane of (z, x), and a couple $-X_1 y_1$ in the plane (x, y). And Y_1 may be replaced by Y_1 at O, a couple $Y_1 x_1$ in the plane of (x, y), and a couple $-Y_1 z_1$ in the plane of (y, z). Therefore the force P_1 may be replaced by X_1, Y_1, Z_1 acting at O, and three couples, of which the moments are, by Art. 201,

$$\begin{aligned} &Z_1 y_1 - Y_1 z_1 \text{ in the plane of } (y, z),\\ &X_1 z_1 - Z_1 x_1 \quad \text{,,} \quad \text{,,} \quad (z, x),\\ &Y_1 x_1 - X_1 y_1 \quad \text{,,} \quad \text{,,} \quad (x, y). \end{aligned}$$

By a similar resolution of all the forces we shall have them replaced by the forces

$$\Sigma X, \quad \Sigma Y, \quad \Sigma Z,$$

acting at O along the axes, and the couples

$$\begin{aligned} &\Sigma(Zy - Yz) = L \text{ suppose, in the plane of } (y, z),\\ &\Sigma(Xz - Zx) = M \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad (z, x),\\ &\Sigma(Yx - Xy) = N \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad (x, y). \end{aligned}$$

Let R be the resultant of the forces which act at O; a, b, c the angles its direction makes with the axes; then by, Solid Geometry,

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2,$$

$$\cos a = \frac{\Sigma X}{R}, \quad \cos b = \frac{\Sigma Y}{R}, \quad \cos c = \frac{\Sigma Z}{R}.$$

Let G be the moment of the couple which is the resultant of the three couples L, M, N; λ, μ, ν the angles its axis makes with the co-ordinate axes; then, by Art. 201,

$$G^2 = L^2 + M^2 + N^2,$$

$$\cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}.$$

206. Problem.—*To find the conditions of equilibrium of any number of forces acting upon a rigid body in any directions.*

A system of forces acting upon a rigid body can always be reduced to a single force and a couple. Since these cannot balance each other, and cannot separately maintain equilibrium, they must both vanish. Hence $R = 0$, and $G = 0$; therefore

$$(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 = 0,$$

and

$$L^2 + M^2 + N^2 = 0.$$

These lead to the six conditions

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0,$$

$$\Sigma(Zy - Yz) = 0, \quad \Sigma(Xz - Zx) = 0, \quad \Sigma(Yx - Xy) = 0.]$$

§ 3. Centre of Gravity.

207. The case in which all the forces acting on a rigid body are parallel to each other and act in the same direction, is a very important one, inasmuch as it includes all ordinary cases where gravity may be considered as the only force acting. For this reason it requires special consideration.

208. First take the case of a weightless rod AB, acted upon by two parallel forces, P, Q, at its extremities. Divide AB in O, so that $\frac{AO}{OB} = \frac{Q}{P}$, and draw OC, OD perpendicular to P and Q. Then, by similar triangles, $\frac{OC}{OD} = \frac{OA}{OB} = \frac{Q}{P}$;

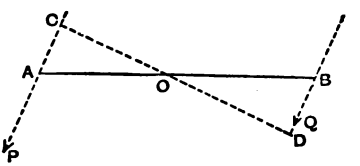

Fig. 19.

$\therefore OC \times P = OD \times Q$, or the moments of P and Q about O are

equal. Hence AB will have no tendency to turn about O, and if it be supported at O by a force P + Q, parallel to P and Q, it will be in equilibrium.

The above proof is quite independent of the inclination of the forces to AB. Conversely, therefore, if AB be acted upon by two parallel forces of constant direction (such as the forces of gravity), it may be turned in any way about the point O, and will still be in equilibrium.

209. The same principle may be extended to any system of parallel forces whatever, as follows.

Theorem.—*Every system of parallel forces has a centre.*

Let W_1, W_2, W_3, W_4 . . . be a rigid system of particles, the forces on which are W_1, W_2, W_3, W_4, . . . respectively. Suppose W_1, W_2 joined by a rigid rod without weight, and divide the same rod in G^1, so that

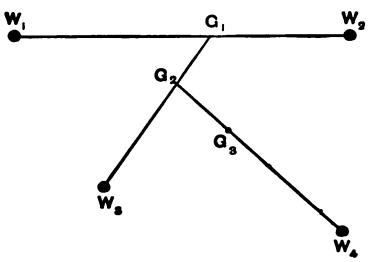

Fig. 20.

$$W_1G_1 : W_2G_1 :: W_2 : W_1 ;$$

then, by Art. 208, W_1 and W_2 will balance in all positions about G_1, and if we suppose G_1 supported, the pressure upon the support will be $W_1 + W_2$.

Again, join G_1W_3, and divide it in G_2, so that

$$G_1G_2 : W_3G_2 :: W_3 : W_1 + W_2 ;$$

then, if we suppose the rod W_1W_2 to rest upon the rod G_1W_3, and G_2 to be supported, the pressure $W_1 + W_2$ at G_1 and W_3 at W_3 will balance about G_2. Hence the three bodies W_1, W_2, W_3, supposed rigidly connected, will balance in all positions about G_2.

Similarly we may find a point G_3 about which W_1, W_2, W_3, W_4 will balance in all positions, and so of any number of particles. Hence every system of parallel forces has a centre.

It is obvious from the method of proof that the system of *particles need not* lie all in one plane.

210. **Theorem.**—*A system of parallel forces can have only one centre.*

For suppose there are two, and let the system be so turned that the two centres lie in the same plane perpendicular to the forces. Then the different forces form a system, the resultant of which must pass through each centre of parallel forces, otherwise the system could not balance about that point; hence the said resultant must pass through two points in the same plane at right angles to its direction, which is absurd. Therefore there are not two centres of parallel forces.

211. **Problem.**—*To find the centre of any number of parallel forces in the same plane.*

Let W_1, W_2, W_3, . . . be the points of application of the forces W_1, W_2, W_3, . . .; in the plane in which they lie, take any rectangular axes, Ax, Ay, at right angles to each other, and let h_1, h_2, h_3, . . . be the distances of W_1, W_2, W_3, . . . from Ax, and k_1, k_2, k_3, . . . their distances from Ay; also, let h, k be the distances of the centre of the system from Ax, Ay respectively; then it is evident that if we find h and k, we shall have solved the problem.

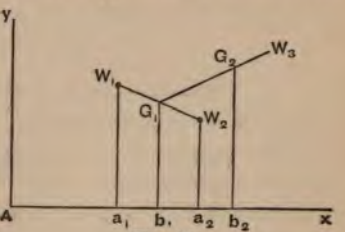

Fig. 21.

Join W_1, W_2, and let G_1 be the centre of parallel forces for W_1, W_2; from W_1, W_2, G_1, draw $W_1 a_1$, $W_2 a_2$, and $G_1 b_1$ perpendicular to Ax; then we have

$$W_1 \times W_1 G_1 = W_2 \times W_2 G_1;$$

but it is evident, from similar figures, that

$$W_1 G_1 : a_1 b_1 :: W_2 G_1 : a_2 b_1;$$

$$\therefore W_1 \times a_1 b_1 = W_2 \times a_2 b_1,$$

$$\text{or } W_1 (A b_1 - k_1) = W_2 (k_2 - A b_1);$$

$$\therefore A b_1 = \frac{W_1 k_1 + W_2 k_2}{W_1 + W_2}.$$

If we consider another force, W_3, we may, in searching for the centre of the three, W_1, W_2, W_3, suppose the two former to act together at their centre, G_1, already found; hence, if G_2 be the centre of the three forces, and we draw $G_2 b_2$ perpendicular to $A x$, we shall have

$$A b_2 = \frac{(W_1 + W_2) A b_1 + W_3 k_3}{W_1 + W_2 + W_3}$$

$$= \frac{W_1 k_1 + W_2 k_2 + W_3 k_3}{W_1 + W_2 + W_3};$$

and so on for any number of forces. Hence we shall have

$$k = \frac{W_1 k_1 + W_2 k_2 \ldots . + W_n k_n}{W_1 + W_2 \ldots . + W_n};$$

and, in like manner,

$$h = \frac{W_1 h_1 + W_2 h_2 \ldots . + W_n h_n}{W_1 + W_2 \ldots . + W_n}.$$

[212. **Problem.**—*To find the centre of any number of parallel forces not in the same plane.*

Take any three planes perpendicular to each other, to form a system of rectangular axes. Let h_1, k_1, l_1, be the perpendicular distances of W_1 from the three planes respectively, and so on of the other particles, and let h, k, l, be the perpendicular distances of the centre; then we may prove, in like manner as in the last proposition, that

$$h = \frac{W_1 h_1 + W_2 h_2 \ldots . + W_n h_n}{W + W_2 \ldots . + W_n};$$

similarly,

$$k = \frac{W_1 k_1 + W_2 k_2 \ldots . + W_n k_n}{W_1 + W_2 \ldots . + W_n};$$

and so likewise,

$$l = \frac{W_1 l_1 + W_2 l_2 \ldots . + W_n l_n}{W_1 + W_2 \ldots . + W_n}\Big].$$

213. We have thus solved the general problem of finding the centre of parallel forces under any circumstances. As already *mentioned, the interest* of this case arises from the fact that the

weight of any body (that is, the attraction of the earth upon its several points or particles) forms such a system of parallel forces. The force in this case having the name of gravity, the centre of parallel forces naturally takes the name of the Centre of Gravity; and in the remainder of this section this name will be used.

214. In discussing the centre of gravity of any body, it is always assumed (unless the contrary is stated) that the particles of the body—that is, the centres of force which it contains—are uniformly distributed throughout its volume. In nature, of course, this is never absolutely true, and the results obtained are therefore approximate only. Also, in speaking of the centre of gravity of any plane figure, such as a triangle, we are really considering that of a thin uniform lamina or slice of that particular form.

215. **Problem.**—*To find the centre of gravity of a physical straight line, or of a uniform thin rectilinear rod.*

The middle point will be the centre of gravity; for we may suppose the line to be divided into pairs of equal weights, equidistant from the middle point, and the middle point will be the centre of gravity of each pair, and therefore of the whole system.

216. **Problem.**—*To find the centre of gravity of a triangle, or of a thin lamina of matter in the form of a triangle.*

Let A B C be the triangle: bisect B C in D, and join A D; draw any line, *b d c*, parallel to B C; then it is evident that this line will be bisected by A D in *d*, and will therefore balance about *d* in all positions; similarly all lines in the triangle parallel to B C will balance about points in A D, and therefore the centre of gravity must be somewhere in A D.

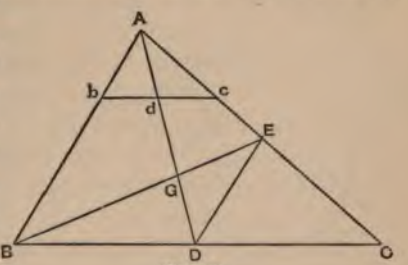

Fig. 22.

In like manner, if we bisect A C in E, and join B E, the centre of gravity must be in B E; hence G, the intersection of A D and B E, *is the centre of* gravity of the triangle.

Join D E, which will be parallel to A B (*Euclid*, vi. 2), then the triangles A B G, D E G are similar.

$$\therefore A G : G D :: A B : D E$$
$$:: B C : D C$$
$$:: 2 : 1,$$
$$\text{or } A G = 2 G D,$$
$$\text{and } \therefore A D = 3 G D.$$

Hence, if we join an angle of a triangle with the bisection of the opposite side, the point which is two-thirds of the distance down this line from the angular point is the centre of gravity of the triangle.

217. Corollary.—*If three equal particles be placed at the angular points of the triangle A B C, their centre of gravity will coincide with that of the triangle A B C.*

For the construction for the centre of gravity will evidently be the same as that just described.

218. Problem.—*Given the centre of gravity of a heavy body, and also that of a certain portion of it, to find the centre of gravity of the remainder.*

Let G be the centre of gravity of the body (Fig. 23), W its weight: G_1 the centre of gravity of the given portion, W_1 its weight. Join G_1G, and in that line produced take G_2, such that

$$G_2G : G_1G :: W_1 : W - W_1.$$

Then G_2 will evidently be the centre of gravity required.

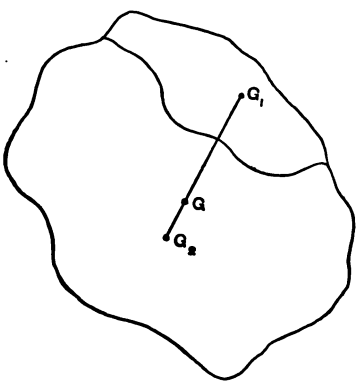

Fig. 23.

The preceding proposition is applicable to a variety of examples: it will enable us, for instance, to find the centre of gravity of the frustum of a *pyramid or of a cone*, that is, the centre of gravity of a portion *of the body cut off* by a plane parallel to the base.

219. The following is one of the most important properties of the centre of gravity.

Theorem.—*When a body is placed upon a horizontal plane, it will stand or fall according as the vertical line through the centre of gravity falls within or without the base.*

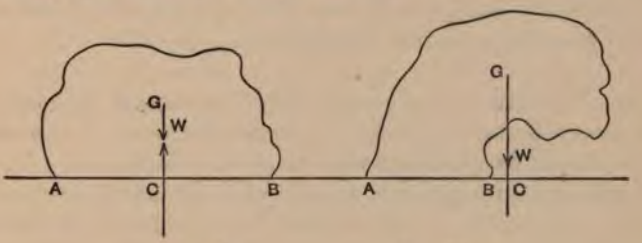

Fig. 24. Fig. 24A.

Suppose the vertical line G C through the centre of gravity G, to fall within the base, as in Fig. 24; then we may suppose the whole weight of the body to be a vertical pressure W acting in the line G C; this will be met by an equal and opposite pressure W from the plane on which the body is placed, and so equilibrium will be produced and the body will stand.

But suppose, as in Fig. 24A, that the line G C falls without the base; then there is no pressure equal and opposite to W at C, and therefore W will produce a moment about B (the nearest point in the base to C), which will make the body twist about that point and fall.

220. According to the proposition just proved, a body ought to rest without falling upon a single point, provided that it is so placed that the centre of gravity is in the vertical line passing through the point which forms the base. And, in fact, a body so situated would be, mathematically speaking, in a position of equilibrium, though practically the equilibrium will not subsist.

This kind of equilibrium, and that which is practically possible, are distinguished by the names of *unstable* and *stable*. Thus an an egg will rest upon its side in a position of *stable* equilibrium, but the position of equilibrium corresponding to the vertical position of its axis is *unstable*. The distinction between stable and

unstable equilibrium may be enunciated generally thus :—Suppose a body or a system of particles to be in equilibrium under the action of any forces; let the system be arbitrarily displaced very slightly from the position of equilibrium, then if the forces be such that they tend to bring the system back to its position of equilibrium the position is *stable*, but if they tend to move the system still farther from the position of equilibrium it is *unstable*.

221. **Theorem.**—*When a heavy body is suspended from a point about which it can turn freely, it will rest with its centre of gravity in the vertical line passing through the point of suspension.*

For let O be the point of suspension, G the centre of gravity, and suppose that G is not in the vertical line through O; draw O P perpendicular to the vertical through G, that is, to the direction in which the weight W of the body acts. Then the force W will produce a moment W × O P about O as a fulcrum, and there being nothing to counteract the effect of this moment equilibrium cannot subsist.

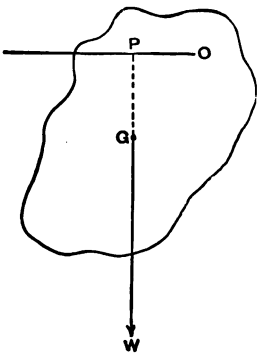

Fig. 25.

Hence G must be in the vertical line through O, in which case the weight W produces only a pressure on the point O, which is supposed immovable.

§ 4. Friction.

222. It nearly always happens that amongst the pressures which keep a body at rest are the reactions of one or more surfaces in contact with each other. Thus, suppose a mass M to rest on a table A B, and suppose it to weigh 1000 lbs.; that weight must be supported by the table, which must *therefore exert* upwards a pressure of 1000 lbs. in a direction

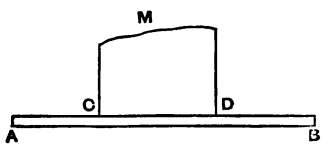

Fig. 26.

opposite to the direction of the weight. This reaction forms what is called a *distributed* pressure, for the under surface, C D, will be in contact with the table at many points, and at each point there will be a reaction. The magnitude of the reactions at the different points is not commonly known; they must, however, be such that their resultant shall act vertically upward through the centre of gravity of M, and shall equal 1000 lbs., otherwise there could not be equilibrium. And, in general, if a body is at rest when pressed against a surface, the various points of that surface must supply reactions whose resultant is equal and opposite to the resultant of the pressures by which that body is urged against the surface. This resultant reaction is called *the reaction* of the *surface.*

223. The amount and the direction of the reaction, which a given surface can exert, depend upon the form of the surface, and the nature of its substance.

224. With regard to its form, a plane surface, such as a table, can exert reaction at an indefinite number of points, as in the last article. But a curved surface, such as a sphere, can only exert reaction through a single point, that, namely, which is in contact with the pressing body. The only exception is where the curvature of the pressing body is the same, but in the reverse direction, as in an ordinary ball-and-socket joint. In this case, as in that of a plane surface, the exact distribution of the reactions is very difficult to determine in practice.

225. With regard to the nature of the surface, let us suppose the forces causing the pressure to be resolved into two components, R and F, the first being perpendicular, and the second parallel, to the reacting surface. It is evident that, if there is equilibrium, there must be reactions, $-R$ and $-F$, equal and opposite to these respectively.

226. With regard to the first, or normal reaction, it is found that however we increase the value of the pressure R, the reaction $-R$ will increase to correspond with it, up to a certain limit. When that limit is reached, the reacting surface gives way, and is either crushed or punctured. The crushing of an egg under a weight is a familiar example of the first; the piercing it with a pin, of the

second. The exact limit for any case depends upon many circumstances connected with the form and nature of the two bodies.

227. With regard to the second, or tangential reaction, this is evidently the same thing as *the resistance opposed by the surface to the motion of a body along it.* To this particular resistance is given the name of Friction.

228. Friction, like the normal reaction, is found to increase, as F increases, up to a certain limit: when this is reached, friction is overcome, and the body slides over the surface. The limit of friction is found to vary very greatly, according to the nature of the surfaces in contact. In some cases it is very large, in others very small. The interposition between the surfaces of some lubricant, such as oil or grease, is found to diminish friction in a remarkable manner. In such cases it is probable that the surfaces are not in contact at all, being divided by a film of the lubricant, over which they slide; and the limit is found to depend greatly on the nature and amount of the lubricant. The laws of statical friction (*i.e.*, of friction between bodies relatively at rest) have been investigated by careful and numerous experiments, especially those of the late General Morin, but the results are only to be considered as general approximations. They have been formulated as follows:—

229. **First Law of Statical Friction.**—*The limit of friction between two bodies at rest is proportional to the normal pressures between the bodies.*

230. **Second Law of Statical Friction.**—*The limit of friction is independent of the area of the surfaces in contact.*

231. Both laws are expressed symbolically by saying that $F = \mu R$, where μ is a number which is constant for any variations of pressure, or of area in contact, and which is approximately the same for all surfaces of the same kind, and under the same conditions as to lubrication, &c. This number is called the co-efficient of friction. It is always less than unity.

232. This equation will necessarily be true, if we suppose that each surface is covered with little hills and hollows, like a file *(Fig. 27), and that* the hills of each fit more or less into the hollows

of the other. The resistance of friction would then be the resistance to the lifting of one surface A away from the other sufficiently to allow the respective hills to pass over each other; and, other things being equal, this resistance would clearly be proportional to the normal pressure. There is no doubt that even the smoothest surfaces are thus roughened; and it is probable that the above explanation accounts for at least the main part of the resistance, in the case of statical friction.

Fig. 27.

233. The laws of friction cease to be even approximately correct when the normal pressure is very great, that is, when it approaches the limit at which the surface would be crushed or punctured, as explained in Art. 226. This is, no doubt, due to the great deformation or alteration of shape which then occurs in the surface.

234. If we suppose μ to vanish, the whole of the reaction between the surfaces becomes normal to the surface. In this case the friction is nothing, and the surface is said to be "absolutely smooth." No such surfaces occur in nature, but it is often convenient to neglect friction, and to treat surfaces as being absolutely smooth. In such cases they are generally called, for brevity, "smooth surfaces." Since $\mu = 0$, we have the principle, that *the reaction of a smooth surface is always normal to the surface.*

235. **Problem.**—*To find the angle at which a rough plane may be inclined, so that a body may just rest upon it without sliding.*

Let α be the angle of inclination of the plane; W the weight of the body; R the normal pressure on the plane; μR the force of friction. Then, resolving the forces perpendicular to the plane and parallel to it, we have these equations of equilibrium:

$$R = W \cos \alpha,$$
$$\mu R = W \sin \alpha;$$
$$\therefore \tan \alpha = \mu.$$

This equation determines the limiting value of the inclination of the plane, for which equilibrium is possible; for any smaller value there will be equilibrium *à fortiori*.

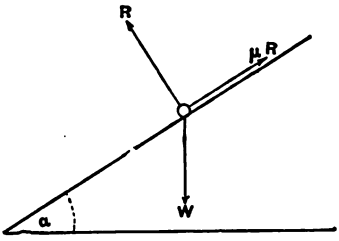

Fig. 28.

236. It will be observed that if the condition which has been investigated be satisfied, equilibrium will subsist however great W may be; the reason being that in whatever manner we increase W, in the same proportion we increase the friction upon the plane, which serves to prevent W from sliding.

237. **Definition.**—*The Angle of Repose for any two surfaces is the angle at which one of the surfaces is inclined to the horizon, when the other, being acted upon by the force of gravity only, just rests upon it without sliding.*

It will be seen by the last article that the tangent of the angle of repose is equal to the co-efficient of friction.

238. **Theorem.**—*The direction of the reaction of a rough surface (when motion is on the point of beginning) is inclined to the perpendicular at an angle equal to the angle of repose.*

Let E be a point of some body in contact with the plane A B, and just prevented from sliding by the friction F.

Let R be the pressure at E in a direction perpendicular to A B, and R′ the resultant of R and F inclined at an angle θ to E R.

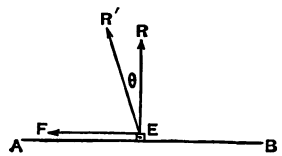

Fig. 29.

Then $R' \sin \theta = F$, $R' \cos \theta = R$,

$$\frac{F}{R} = \frac{R' \sin \theta}{R' \cos \theta} = \tan \theta,$$

But $\frac{F}{R} = \mu$; therefore $\mu = \tan \theta$,

But $\mu = \tan \alpha$,

therefore $\theta = \alpha$,

i.e., the direction of the resultant reaction, or R′, is inclined at an angle α to the perpendicular.

§ 5. Virtual Velocities.

239. Let us suppose a body, or system of particles, to be acted on by a certain number of external forces which are in equilibrium among themselves. Further, suppose the body, instead of being actually at rest, to have a motion of its own; but a motion such that its change of position does not affect the amount or the relations of the external forces, so that, at the end of any given displacement, these are the same as they were at the beginning. Such a displacement may be called a Conservative Displacement. For a displacement to be conservative, any rigid body in the system must retain its exact form, any two bodies in contact must remain in contact, any rods or strings in the system must remain unbroken, and so forth.

240. **Theorem.**—*If the external forces acting on a system are in equilibrium, and if the system receive a conservative displacement, the net energy exerted by the external forces will be zero.*

The net energy exerted by the forces is the same thing as the net effect produced by the forces during the displacement (Art. 136). Now if this net effect be not zero, it must be an effect tending either to increase or to retard the motion of the system. But, by the principle of symmetry, if the forces are in equilibrium, they must have no more tendency to increase motion than to retard it. Hence the net energy exerted by the forces must be zero.

The same thing will follow from considering that, if the forces could produce an accelerating effect by any displacement, they would of themselves cause that displacement, and therefore they could not be in equilibrium.

241. **Definition.**—*If the point of application of a force be supposed to be displaced through an indefinitely small space, then this hypothetical displacement, as measured in the direction of the force, is called the virtual velocity of the force; and the product of the force and the virtual velocity is called the virtual moment.*

Thus, let P be the force, A the original, and B the new point of application, and draw BN perpendicular to P. Then AN is the virtual velocity, and P × AN is the virtual moment.

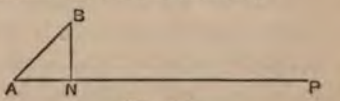

Fig. 30.

242. The terms virtual velocity and virtual moment are here retained, as they are used in many works on Mechanics. But it will be seen that the expression for the virtual moment is the same thing as that for the energy exerted by the force during the displacement (Art. 129). Hence, by the last theorem, Art. 240, we have the following principle :—*If a system in equilibrium receive a very small conservative displacement, the sum of the virtual moments of all the external forces (or the sum of the products of the forces and their virtual velocities respectively) is zero.*

243. This principle is generally known as the principle of Virtual Velocities, and is of very great use in the solution of practical problems in Statics. It may be proved independently of the conception of energy; but the proof is cumbrous and difficult, and is quite needless, if the principles of energy have once been explained.

244. It will be observed that, in Art. 240, the limitation to an indefinitely small displacement was not made. In many cases (as will be seen farther on, Arts. 295, 300) a finite displacement may be conservative, and it may even be more convenient to consider a finite displacement. In many other cases any finite displacement is sufficient to alter the amount or direction of the forces, so as to prevent the principle of Art. 242 from being applicable. This difficulty can always be got over in practice by supposing the displacement to be indefinitely small. For, let P be any force, and p its virtual velocity under any displacement; and let the effect of the displacement be to alter P to $(P + dP)$. Then the true value of the virtual moment would be $(P + dP)\,p$. But if p be exceedingly small, dP will be exceedingly small, and therefore their product, $p\,dP$, will be exceedingly small as compared with the other term $p\,P$. Hence the product $p\,dP$ may be neglected without sensible error, and we may use the principle of virtual velocities as if P remained absolutely unaltered.

245. One special advantage of this principle is that it enables us, in forming the equations of equilibrium for any given problem, to omit altogether a number of forces which would otherwise have to be included. Such forces evidently are :—

(1.) Forces acting at a point in the system which remains at

rest during the displacement: for their virtual velocities are zero.

(2.) Equal and opposite forces, such as the reactions between two surfaces in contact: for their virtual moments are also equal and opposite, and cancel each other.

(3.) Forces whose directions are at right angles to the displacements of their points of application: for their virtual velocities are zero.

§ 6. Machines in General.

246. A machine is a mechanical contrivance for enabling power to be applied in the most convenient way for the doing of a certain required work. The power may be that of men, of horses, of the steam-engine, &c.; and the work may be of any kind, such as the drawing of a cart, the lifting of coals from a pit, the planing of a piece of metal, &c. In all such cases there is some resistance to be overcome: and as the resistance of gravity is the most common and familiar form of resistance in practical work, the name of the Weight has been commonly given to the resistance, the overcoming of which constitutes the work to be done. The force exerted by the power is also called, for shortness, the Power; and the general problem with regard to machines may be stated as being to find the relation between the power and the weight. Sometimes it is most convenient that this relation should be one of equality: in other words, that the force applied by the power should be equal to the force applied directly to the weight. The common L-shaped crank, used in bell-hanging to alter the direction of the wires, is a good example of this. Generally, however, it is most convenient that the power should be very different from the weight. Thus, if a man has to lift a weight of 1 ton hanging by a rope, it is clear that he cannot do it unless the pulleys, or other mechanical contrivance provided, enable him to lift the weight by exercising a pull of, say, 1 cwt. When, as in this case, which is a very common one, the power is smaller than the weight, mechanical advantage is said to be gained. When, as in some other cases, it is desirable that the power should be greater than

the weight, mechanical advantage is said to be lost. A machine, therefore, may be described generally as an apparatus for obtaining mechanical advantage.

247. **Theorem.**—*Whenever, in a machine, the power and the weight may be taken as the only forces acting, then their amounts are inversely proportional to the displacements of their points of application.*

Let P be the power, W the weight, and let all other forces be left out of account in applying the principle of virtual velocities. Let p be the displacement of P, and w of W, in any small conservative displacement of the system. Then the resulting equation must be

$$P\,p = W\,w, \text{ or } \frac{P}{W} = \frac{w}{p}.$$

Hence the smaller P is in comparison with W, the smaller w will be in comparison with p. But the smaller P is in comparison with W, the greater is the mechanical advantage gained. Hence we arrive at the important principle, that the greater the mechanical advantage gained, the less will be the displacement of the weight in comparison with that of the power. Since, if motion actually takes place, the velocities of the points of application will be in proportion to their displacements, this principle may be put in another form, in which it is usually known—viz., *Whatever is gained in power is lost in speed.*

248. There are no cases in which the assumption just made, namely, that the weight and power are the only forces to be considered, rigorously holds. In every movement of a machine there will always be a certain amount of friction; and friction can never be left out of the equation of virtual velocities, because it always acts along the line in which the point of application moves. There are, however, cases (as that of a balance resting on a knife-edge) where the friction is exceedingly small; and for these the principle, that what is gained in power is lost in speed, is very approximately true. Where the friction is considerable this is no longer the case.

249. If we include friction, the equation for any machine will take the form,

$$P\,p - W\,w + F\,f = 0,$$

where F is the friction, f its virtual velocity. In general, if we attempt to make p very large, and thus get P very small, we introduce complexities, which increase the value of F; and this puts a practical limit to the mechanical advantage which it is possible to obtain by the use of machines.

250. The name of Mechanical Powers has been given to certain simple contrivances for obtaining mechanical advantage, which are constantly used, and form the elements, as it were, out of which more complicated machines are built up. They are generally enumerated as follows:—The lever, the wheel and axle, the toothed wheel, the pulley, the inclined plane, the wedge, and the screw. In addition to their importance as elements of machinery, they form very instructive examples of the application of the principle of virtual velocities; and they will therefore be considered separately.

§ 7. The Lever.

251. The conditions of equilibrium of the lever, considered as a system under three forces, have already been discussed (Art. 186); but its great importance, forming as it does the type to which almost all other contrivances may be ultimately reduced, requires that we should give it a fuller description.

252. Problem.—*To prove the principle of the lever by the method of virtual velocities.*

Let A B be the lever (Fig. 31), having its fulcrum at O, and let P, Q be the forces acting at A and B. Draw O D, O E perpendicular to the directions of P and Q. Suppose the lever to be turned round the fulcrum O through a very small angle, $d\theta$, into the position $a\,b$. Then the total distance through which A will have moved will be represented by $OA \times d\theta$; and the distance measured in the direction of P will be

$$OA \times d\theta \times \cos(a\,A\,D) = OA \times \sin(DAO) \times d\theta = OD \times d\theta.$$

Similarly, the displacement of B, measured in the direction of Q, will be $OE \times d\theta$.

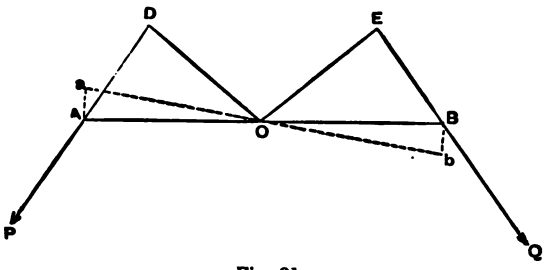

Fig. 31.

Hence, if we form the equation of virtual velocities, it will be as follows :—

$$P \times OD \times d\theta = Q \times OE \times d\theta,$$
$$\text{or } P \times OD = Q \times OE.$$

This is the principle of the lever, as previously demonstrated, Art. 189.

In the above, we have supposed friction to be neglected; and if the lever turns about a sharp edge, like the scale beam of a balance, the friction will really be exceedingly small. In machines, however, the levers usually consist of flat bars, turning about rounded pins or studs, which form the fulcrums; and between the lever and the pin there will, of course, be friction. It is easy to form the equation of equilibrium in this case. Let us call P the power, Q the weight; and let us make the usual assumption that P is about to overcome Q, so as to make the lever turn in its own direction. Then the friction will act as against P. To find the amount of the friction, let us remember that the pressure on the fulcrum must be equal to the resultant of P and Q—say R. The resistance of friction (Art. 231) will be found by multiplying this by the coefficient of friction, or μ. Hence the amount of the friction is μR, and it acts tangentially to the surface of the pin. Let r be the radius of the pin; then the displacement of the point of application of the friction, by the turning of the lever through the angle $d\theta$, will be $r\,d\theta$. Hence the virtual moment of the friction will be

$$\mu R \times r\,d\theta.$$

Hence the equation of virtual moments will now be formed as follows :—

$$P \times OD = Q \times OE + \mu R r.$$

253. **Cor.** Let p, q be the lengths of the two arms of a straight lever, and let P, Q be parallel to each other. Then $R = P + Q$, and the equation becomes

$$P p = Q q + \mu (P + Q) r,$$
$$\text{or } P (p - \mu r) = Q (q + \mu r).$$

254. We shall give yet another proof of the principle of the lever, which is independent of the parallelogram of forces, being derived directly from the principle of symmetry. From this principle it follows that a uniform heavy rod will balance about its middle point, since there is no reason for its turning in one direction rather than the opposite Therefore, an upward pressure applied at the middle point, and equal to the weight, would keep the rod at rest; and therefore the statical effect of the rod is to cause a downward pressure at the middle point equal to its weight.

This being premised, let A B be a heavy uniform rod equal in weight to the sum of two given weights, P and W; then the rod A B balances about its middle point C.

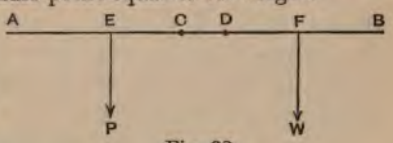

Fig. 32.

Divide A B in D, so that A D : D B :: P : W, and let E be the middle point of A D, F of D B; then A D or P may be conceived to be collected at E, and B D or W at F. Consequently P acting at E will balance, about the point C, W acting at F.

But

$$EC = AC - AE = BC - ED = DB - EC;$$
$$\therefore DB = 2EC;$$

similarly

$$AD = 2CF;$$
$$\therefore P : W :: AD : DB,$$
$$:: CF : EC;$$
$$\text{or } P \times EC = W \times CF.$$

255. Upon the doctrine of the lever thus demonstrated, it is possible to construct a complete system of statics; the steps are as follows. From the preceding proposition it is easy to conclude that two forces, acting at the extremities of the arms of any lever, will produce equilibrium when their moments are equal and tending to twist in opposite ways. And this being premised, we can demonstrate the parallelogram of forces.

Let A m, A n represent in magnitude and direction two forces P and Q acting at the point A; complete the parallelogram A m B n, and join A B. Also draw B C, B D perpendicular to A m, A n produced. Now suppose A B to be a lever movable about B, and acted on by the forces P and Q at A. Then

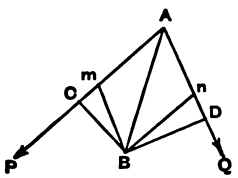

Fig. 33.

$$\frac{P}{Q} = \frac{A\,m}{A\,n} = \frac{\sin(m\,B\,A)}{\sin(m\,A\,B)} = \frac{\sin(n\,A\,B)}{\sin(m\,A\,B)} = \frac{B\,D}{B\,C},$$

or $$P \times B\,C = Q \times B\,D;$$

therefore the forces P and Q would keep the lever at rest.

And since the resultant of P and Q would produce the same effect as P and Q together, it also acting at A would keep the lever at rest. But no single force acting at A can keep the lever at rest, unless it act in the direction A B; consequently, A B is the *direction* of the resultant.

That A B represents the resultant in *magnitude* must be proved as in Art. 117.

256. Levers are frequently divided, for convenience, into three classes, according to the position of the fulcrum.

257. The first class has the fulcrum between the power and the weight. In this case any amount of mechanical advantage may be gained, by making the arm upon which the power acts sufficiently long. A crowbar used to lift great weights, a poker, a pair of scissors, are examples. Thus, with the poker, the coals are the weight, the bar of the fireplace the fulcrum, the force exerted by the hand the power.

258. The second class of lever has the weight between the fulcrum *and the power.* The oar of a boat is an example, in which the

water forms the fulcrum, the resistance of the boat applied at the rowlock the weight, and the power is applied by the hand of the rower. In this case mechanical advantage is gained.

259. The third class has the point of application of the power between the fulcrum and the weight. The most interesting example is the human arm, when applied to lift a weight by turning about the elbow; here the fulcrum is the elbow, and the power is applied at the wrist by means of sinews, which exert a force when the muscles of the arm contract. In this case mechanical advantage is lost, but this is more than made up by the quickness of action obtained.

260. In all the cases here mentioned, and in the mechanical powers generally, the object is to produce motion, not rest; in other words, to do work. The statical investigation of these cases shows only the limit of force to be applied before any work can be done. For any useful effect, the force applied must exceed this limit, and the greater the excess, the greater the amount of work done. There is, however, one class of applications of the lever where the object is not to do work, but to produce equilibrium, and which are therefore specially adapted for treatment by statics. This is the important class of measuring machines, where the object is not to overcome a particular resistance, but to measure its amount. A testing machine is a good example, measuring the pull which a bar of any material will sustain before breaking. But the most familiar examples are weighing machines. These are of two kinds—(1.) those in which the arm is constant, but the weight varies, or balances; (2.) those in which the weight is constant, but the arm varies, or steelyards.

261. We proceed to describe the common balance.

Let A B be a rigid rod, C D a small rigid piece attached to its middle point and perpendicular to it, and let D be fixed; E, F two scales or pans of equal weight depending from the extremities A and B. Then it is evident that if the scales be equally loaded, the *beam* A B will be horizontal; and if not, that the more heavily loaded scale will cause the extremity to which it is attached to pre-

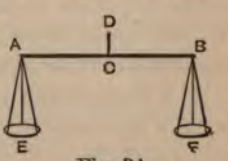

Fig. 34.

ponderate. For, if B sinks, turning about D, the moment of F about D will diminish, while that of E increases; and this will go on till the two moments become equal, which will be the position of equilibrium, if the beam have no weight. The weight of the beam also tends to produce equilibrium, since its moment is in the opposite direction to that of F. For the sake of clearness of explanation we have spoken of C D as a small piece attached to A B; in reality D is merely the point of suspension of the beam to the extremities of which the scales are attached. In practice, the weight of the beam may be used alone to balance the excess in one scale; and then the points D and C may coincide.

262. The preceding explanation represents the balance in its simplest form, and exhibits its principles; in practice many modifications and additional contrivances must be introduced. Much skill has been expended upon the construction of balances, and great delicacy has been obtained. Thus, the beam should be suspended by means of a knife edge, that is, a projecting metallic edge transverse to its length, which rests upon a plate of agate or other hard substance. The chains which support the scales should be suspended from the extremities of the beam in the same manner. The point of support of the beam should be at equal distances from the points of suspension of the scales; and when the balance is not loaded the beam should be horizontal.

263. To test the accuracy of a balance, first ascertain that the beam is horizontal when the balance is not loaded: then place two weights in the scales such that the beam shall be horizontal: lastly, change these weights into opposite scales; then if the beam still remain horizontal the balance is a true one.

264. The chief requisite of a good balance is what is termed *sensibility;* that is to say, if two weights which are very nearly equal be placed in the scales, the beam should vary *sensibly* from its horizontal position. In order to produce this result two conditions should be satisfied.

(1.) The point of support of the beam and the points of suspension of the scales should be in the same straight line. The consequence *of this will* be that two equal weights in the scales will

produce a resultant through the point of support; they will therefore have no effect whatever in twisting the beam, and the deviation from horizontality will be the same for a given *difference* of weights, however great the weights themselves may be.

(2.) The point of support should be very near the centre of gravity of the beam, and a little above it. The nearer these two points are to each other the greater will be the sensibility, for the weight of the beam acting at its centre of gravity must in this case be in equilibrium with the small difference of the weights, acting at one end of the beam, and this difference of the weights will act at a greater mechanical advantage the nearer the centre of gravity of the beam is to the fulcrum.

265. These facts may be examined analytically, by forming the equations for a balance when in equilibrium. Let A C B (Fig. 35) represent such a balance. Let E, F be the weights in the scales, and let G be the weight of the beam itself. Let the angle $D A C = \alpha$, $C A E = \beta$; then $C B F = \pi - \beta$. Then if we take $A D = c$, $D C = a$, $C A = d$, the equation of moments about D may be written

$$F c \sin(\beta - \alpha) = E c \sin(\beta + \alpha) + G a \cos \beta.$$

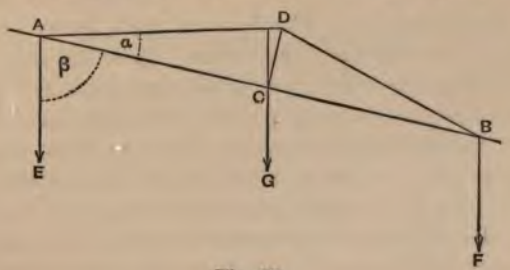

Fig. 35.

From this equation β may be calculated when the weights are known: or conversely, if β be observed, the difference between the weights F and E may be calculated.

If $\alpha = 0$, the equation becomes

$$(F - E) c \sin \beta = G a \cos \beta,$$

a being now the distance of the centre of gravity of the beam below D. Hence it follows that β diminishes as a diminishes, or the nearer the centre of gravity to the centre of suspension, the greater will be the sensibility.

266. We will now proceed to the other form of weighing machine, or the steelyard.

If a beam A B (Fig. 36) is suspended about an axis passing through its centre of gravity C, and on the arm B C is placed a movable weight G, then if a body Q, equal in weight to G, is suspended from A, the beam will balance when G is at a distance from C equal to A C. If the body Q equals twice the weight of G, the beam will balance when G's distance from C equals twice A C; and so on in any proportion. The beam is made heavy at the end A, so that C is very near that point; then the arm B C can be divided into *equal* divisions, which indicate the weight of a substance suspended at A by means of the position occupied by G when it balances that substance. An instrument constructed on this principle is called a steelyard, and is used when heavy substances have to be weighed, and extreme accuracy is not required; the advantage it possesses arises from the fact that the weights employed are much less heavy than the substance to be weighed.

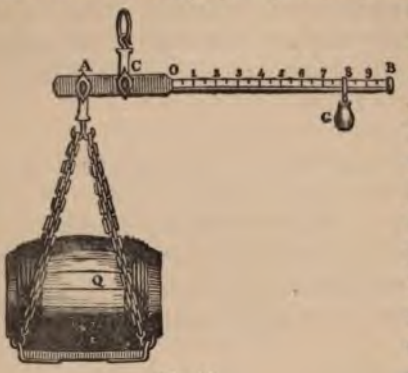

Fig. 36.

267. If the point of suspension is not coincident with the centre of gravity, the graduation will be performed as follows:—

Problem.—*To graduate the common steelyard, when the fulcrum is in any position.*

Let C be the fulcrum (Fig. 37), W the substance to be weighed,

hanging at the extremity A, P the movable weight. Now if the weights W and P were removed, the longer arm of the steelyard, which is that upon which P hangs, would preponderate; suppose then that B is a point such that P hanging from it would keep the steelyard in a horizontal position, and take C D = B C. Then the moment about C, produced by the weight of the steelyard itself, is equivalent to the moment of P hanging from D.

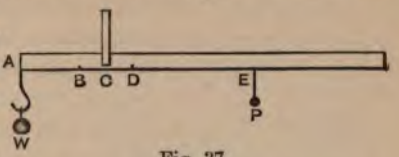

Fig. 37.

Now, let W hang from A, and P from any point E, then for equilibrium we must have

$$W \times AC = P \times CD + P \times CE = P \times BE;$$

$$\therefore BE = \frac{W}{P} \times AC.$$

Suppose that P = 1 lb., and make W successively equal to 1 lb., 2 lbs., 3 lbs., &c., then the values of B E will be A C, 2 A C, 3 A C, &c., and these distances must be set off, measuring from B, and the points so determined marked 1 lb., 2 lbs., 3 lbs., &c.

§ 8. The Wheel and Axle.

268. This machine consists of two cylinders having their axes coincident (Fig. 38), the two cylinders forming one rigid piece, or being cut from the same piece; the larger is called the wheel, the smaller the axle. The cord by which the weight is suspended is fastened to the axle and coiled round it; the power acts sometimes by a cord coiled round the wheel, sometimes by handspikes, as in the *capstan*, sometimes by handles, as in the *windlass*.

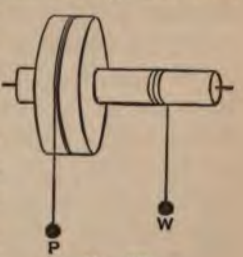

Fig. 38.

269. **Problem.**—*To find the ratio of the Power to the Weight, when there is equilibrium upon the Wheel and Axle.*

Let A B, C D (Fig. 39) be the wheel and axle, having the common centre O; P and W the power and weight, supposed to act by strings at the circumference of the wheel and axle respectively.

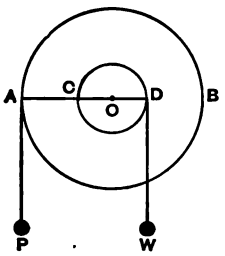

Fig. 39.

For simplicity's sake P, W, and the arms at which they act are represented in the figure as in the same plane.

From the common centre O draw O A, O D to the points at which the cords supporting P and W touch the circumferences of the wheel and axle respectively; these lines will be perpendicular to the directions in which P and W act; hence, by the principle of moments,

$$P \times A\,O = W \times O\,D,$$

$$\text{or } \frac{P}{W} = \frac{O\,D}{A\,O} = \frac{\text{radius of axle}}{\text{radius of wheel}}.$$

270. It is evident that, by increasing the radius of the wheel, any amount of mechanical advantage may be gained. It will also be seen that the principle of the wheel and axle is merely that of the lever; the peculiar advantage of the wheel and axle being this, that an endless series of levers (so to speak) are brought into play, which is essential to the practical use of the lever, when applied to such purposes as raising a bucket in a well, heaving an anchor, or the like.

271. In the above investigation we have neglected friction, or, which comes to the same thing, have assumed that the point on which the axle turns is indefinitely small. In practice, of course, the pivot has a certain radius r, and a certain coefficient of friction μ. Then, the pressure on the pivot is $P + W + w$ (where w is the weight of the apparatus), and the friction is $\mu\,(P + W + w)$ acting tangentially to the radius r. If we suppose P to be about to overcome W, this force will act in opposition to P. Hence, the true equation of moments will be

$$P + A\,O = W \times O\,D + \mu\,(P + W + w)\,r.$$

§ 9. The Toothed Wheel.

272. Toothed wheels are wheels provided on the rim with projections or teeth, which interlock, as shown in Fig. 40, and which are therefore capable of transmitting force. When the teeth are shaped correctly, the wheels will roll upon one another, just as two ideal circles, indicated by the curved lines *a b c*, *d e c*, and called *pitch circles*, will roll together. The pitch circle of a toothed wheel is thus the element which determines its value in transmitting motion. Suppose that two axes at a distance of 10 inches are to be connected by wheel-work, and are required to revolve with velocities in the proportion of 3 to 2. Now, two circles, centred upon the respective axes, and having radii 4 and 6 inches, would by rolling contact move with the desired relative velocity: hence these must be the pitch circles of the wheels when made. Thus, whatever may be the forms of the teeth upon the wheels to be constructed, the pitch circles are determined beforehand.

Fig. 40.

273. The curves to be given to the teeth in order that the wheels when made shall run truly upon one another are described in Goodeve's *Elements of Mechanism* and elsewhere. The direction of the transmission of force between the wheels is not absolutely fixed in direction, but the teeth commonly used are so shaped that the main part of the action takes place in a line touching the two pitch circles at their point of contact.

274. It follows from the above, that in treating of toothed wheels, we may treat them as if they were reduced to their pitch circles, rolling against each other. Suppose that A B C, A D E (Fig. 41) are two such circles, in contact with each other at A, and that the power P acts at C, and the weight W at A. Then, if we suppose the wheels to turn through a small angle, their peripheries must move through the same space, and therefore the displacement of P will be the same as that of W. Hence, by the principle of virtual

velocities, P must = W, or no mechanical advantage is gained. Motion is often transmitted in such a manner, for the purpose of gaining space, but no mechanical advantage is gained thereby.

If, however, we place a smaller wheel, O F G, on the axis M of A B C, and a larger wheel, O H K, on the axis N of A D E, and then slip A B C and A D E past each other, so as to make O F G and O H K gear at O, then the case is altered. The wheels, A B C and O F G, then form a wheel and axle, and a tangential pressure is given at O, the value of which is $P \times \frac{A M}{O M}$. The wheels, O H K and A D E, also form a wheel and axle, of which the power is represented by the pressure $P \times \frac{A M}{O M}$ at O, and the weight by W at A. Hence, by Art. 269, we have

$$W = P \times \frac{A M}{O M} \times \frac{O N}{A N}.$$

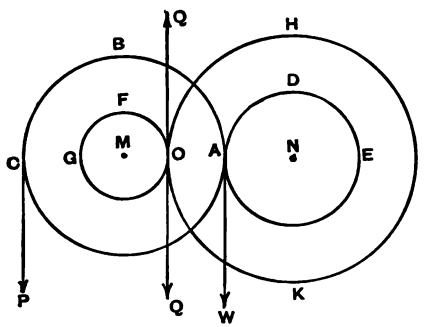

Fig. 41.

275. In such an arrangement, the larger wheel of each pair is called the wheel, and the smaller the pinion. Any number of such pairs may be employed in succession, and form what is termed a Train of Wheels. If R_1, R_2, R_3 . . . be the radii of the wheels in such a train, and r_1, r_2, r_3 . . . those of the pinions, it is evident that the relation of power to weight will be given by

$$\frac{P}{W} = \frac{r_1 \times r_2 \times r_3 \times \ldots}{R_1 \times R_2 \times R_3 \times \ldots}$$

276. In the above we have not taken any account of friction. It is clear that in this case there will be two sorts of friction :—(1.) the *friction of the* bearings in which the axle moves, and (2.) the

friction of the teeth (as at O), which cannot move past each other without a certain amount of grinding.

The amount of the first, or axle friction, will be proportional to the pressure on the bearing, and it will act at an arm ρ, where ρ is the radius of the bearing.

The amount of the second will be proportional to the pressure between the teeth, and if this be called Q, we may take the friction to be m Q. It will act at an arm which will be nearly equal to the radius of the wheel, and may be taken as exactly equal to it.

Taking two pairs of wheels and pinions only, and following the notation of the last article, the equation of moments for the first pair will be

$$P \times R_1 = Q \times r_1 + m Q \times r_1 + \mu (Q + P) \rho_1.$$

The equation for the second pair will be

$$Q \times R_2 = W \times r_2 + \mu (Q - W) \rho_2.$$

Combining the two equations, we have

$$P \times (R_1 - \mu \rho_1) = (r_1 + m r_1 + \mu \rho_1) \times \frac{W(r_2 - \mu \rho_2)}{R_2 - \mu \rho_2},$$

which gives the proportion between P and W.

The weights of the wheels, &c., themselves are here neglected.

277. It will be observed, in toothed gearing, that the smaller the radius of the pinion as compared with the wheel, the greater will be the mechanical advantage gained. There is, however, a practical limit to the size that can be given to the pinion, because the teeth must be of a certain size for strength, and must not be too few in number. Six is generally the least number admissible for the teeth of a pinion. They are often joined together by a solid rim at each end, beyond the teeth of the wheel. Such a pinion is called a lanthorn pinion.

278. The equation, Art. 275, shows that by a train consisting of a very few pairs of wheels and pinions, an enormous mechanical advantage may be gained. Thus, suppose there are three pairs, and that the ratio of the wheel to the pinion in each is 10 to 1. Then the ratio of P to W is $\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = \frac{1}{1000}$. Thus P is only

one thousandth part of W; but on the other hand, W will only make one turn where P makes one thousand. Such trains of wheels are very useful in machines such as hand cranes, where it is not essential to obtain a quick motion, and where the power available is very small in comparison to the weight.

§ 10. The Pulley.

279. The pulley, in its simplest form, consists of a wheel, capable of turning about its axis, which may be either fixed or movable. A

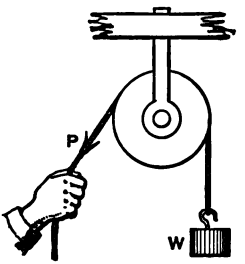

Fig. 42.

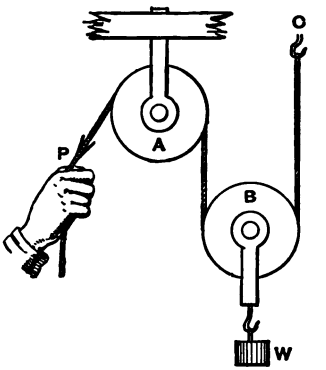

Fig. 43.

cord passes over a portion of the circumference; if the axis of the pulley is fixed (Fig. 42) its only effect is to change the direction of the force exerted by the cord, but if it is movable, as at B (Fig. 43), a mechanical advantage may be gained, as we shall see immediately. Combinations of pulleys may be made in endless variety; we shall here consider only the simple movable pulley, and some of the more ordinary combinations.

280. No account will here be taken of the weights of the pulleys themselves, or of friction, or of the weight and rigidity of the cords. With regard to the first, the weight of a set of pulleys *is generally* small in comparison with the loads which they lift; *and the friction* also need not be large. The stiffness of the cord,

if the pulleys are small, is not inconsiderable; but to introduce it would complicate the problem too much. We therefore, in this and similar cases, treat the cord as a "string;" by which is meant in mechanics a thin, weightless, flexible line, simply serving to transmit tensile force along its length. The force which a string exerts has no component at right angles to its direction when in equilibrium. It is therefore incapable of opposing any resistance to a force acting at right angles to its direction; it simply takes up the position which such a force assigns to it, without modifying its tension in any way. Hence, if a string pass over any smooth surface, its tension is unaltered; for all the reactions of such a surface are normal to the direction. This is sometimes expressed by saying that the tension of a string is the same throughout; but this is misleading, if used without qualification, because it does not hold for points where the string is acted upon by forces, such as friction, which have components parallel to its direction.

281. **Problem.**—*To find the ratio of the power to the weight in the single movable pulley.*

Let O be the centre of the pulley, which is supported by a cord passing under it and attached to some fixed point C at one end, and stretched by the force P at the other. Suppose the weight W to be suspended from the centre O.

Join the points A, B, at which the contact of the cord with the pulley commences, by a straight line A B, which will pass through the centre O. Then we may consider the mechanical conditions of the problem to be the same as those of a lever A B, kept in equilibrium about the fulcrum O by the force P at A and the tension of the string at B. But the tension of the string, neglecting friction, must be the same throughout, and is therefore equal to P. Hence the force at each end of the lever is P, and the resultant of these two parallel forces 2 P.

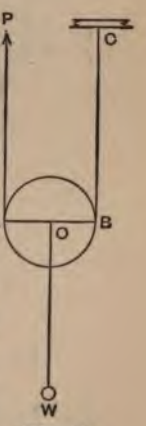

Fig. 44.

But this resultant supports W;

$$\therefore\ 2P = W,$$

$$\text{or } \frac{P}{W} = \frac{1}{2}.$$

282. The same result follows by the principle of virtual velocities. For suppose the pulley, and the weight W with it, to be lifted by any distance. Then both the halves of the string, C B and A P, must be shortened by the same distance, and hence P must move through double the distance. In other words, P's displacement is double W's displacement, and therefore $P = \frac{1}{2}$ W.

283. **Problem.**—*To find the ratio of the power to the weight, in a system of pulleys in which each pulley hangs by a separate string.* (First system of Pulleys.)

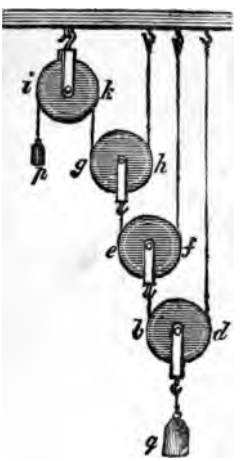

Fig. 45.

This system of pulleys is represented in Fig. 45. Suppose there are n pulleys; then the tension of the string passing under the first $= \frac{W}{2}$ (by the property of the simple pulley). The tension of the string passing under the second $= \frac{W}{2^2}$, and so on. That of the string under the last pulley $= \frac{W}{2^n}$; but this must be equivalent to the power P;

$$\therefore P = \frac{W}{2^n},$$

$$\text{or } \frac{P}{W} = \frac{1}{2^n}.$$

284. **Problem.**—*To find the ratio of the power to the weight, in a system of pulleys in which the same string passes round all the pulleys.* (Second system of Pulleys.)

This system is represented in Fig. 46. There are two blocks,

the lower one movable, and each containing a number of pulleys. Since the same string goes round all the pulleys, the tension throughout will be the same, and equal to the power P. Let n be the number of strings at the lower block, then the sum of their tensions will be n P, and we shall have

$$n\,\mathrm{P} = \mathrm{W},$$

$$\text{or } \frac{\mathrm{P}}{\mathrm{W}} = \frac{1}{n}.$$

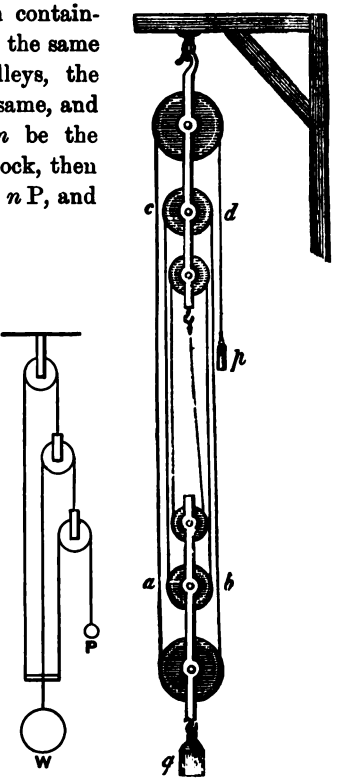

Fig. 47.

285. **Problem.**—*To find the ratio of the power to the weight, in a system of pulleys in which all the strings are attached to the weight.* (Third system of Pulleys.)

Fig. 46.

Fig. 47 represents the system. The tension of the string by which P hangs is P; that of the next = 2 P (by the property of the simple pulley); that of the next 2^2 P, and so on. Let there be n strings, then the tension of the last = 2^{n-1}P, and the sum of all the tensions

$$= (1 + 2 + 2^2 + \ldots\ldots\ldots + 2^{n-1})\ \mathrm{P} = \mathrm{W};$$

$$\text{or } \frac{\mathrm{P}}{\mathrm{W}} = \frac{1}{1 + 2 + 2^2 + \ldots\ldots\ldots + 2^{n-1}} = \frac{1}{2^n - 1}.$$

286. The second system is the only one of practical importance. In practice the pulleys are made all of the same size, and placed side by side in one frame, the rope being inclined slightly aside to pass from one pair to the next. This forms what is called a set of

Blocks and Falls, which is very commonly used on shipboard, and wherever weights have to be lifted at irregular times and places.

The weight of the lower set of pulleys in this case merely forms part of the gross weight W.

287. The friction on the spindle of any particular pulley is proportional to the total pressure on the pulley, which is clearly 2 P. It may therefore be taken as $\mu \times 2$ P, and the amount of its displacement, when W is raised, will be to the displacement of W in the ratio of the radius of the spindle, r, to that of the pulley, R.

§ 11. The Inclined Plane.

288. In the mechanical powers, as hitherto considered, the friction is mainly that of lubricated bearings, and is relatively small. In the three which remain—the Inclined Plane, Wedge, and Screw—the friction is, however, considerable, and must not be neglected.

289. **Problem.**—*To find the force required to overcome the resistance of a heavy body on an inclined plane.*

Let AB be the plane (Fig. 48), α its inclination to the horizon.

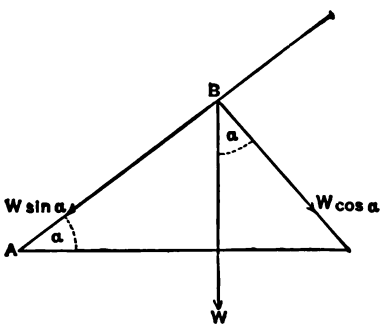

Fig. 48.

Let W be the weight of the body acting at B. Resolve this into two components, parallel and perpendicular to the plane. The value of these will be W sin α and W cos α respectively. The former offers a direct resistance to movement up the plane, and the latter occasions a resistance of friction μ W cos α, which acts in the same direction. Hence the force P, which, acting along the plane, will just overcome the *resistance of* W to upward movement, is given by

$$P = W \sin \alpha + \mu W \cos \alpha.$$

290. **Cor. 1.** If P does not act along the plane, but makes an angle β with it, we may resolve it into components, P cos β and P sin β, parallel and perpendicular to the plane respectively. The latter goes to diminish the component W cos α, and therefore the force of friction, while the former acts directly to overcome the resistance. Hence the equation in this case is

$$P \cos \beta = W \sin \alpha + \mu (W \cos \alpha - P \sin \beta),$$

or

$$P (\cos \beta + \mu \sin \beta) = W \sin \alpha + \mu W \cos \alpha.$$

291. **Cor. 2.** If we suppose the friction to be zero, the relation between P and W is given by

$$P = W \sin \alpha.$$

292. It follows from Cor. 2, that the smaller the inclination to the horizon, the greater will be the mechanical advantage gained. If, however, we take in friction, this is no longer necessarily the case, for the term $\mu W \cos \alpha$ increases as α diminishes.

The gradients on railways are the most common examples of the use of the inclined plane; these are always made as low as is convenient, in order to enable the engine to lift the heaviest possible train. Is there, however, a limit to the advantage thus gained? We will consider this question.

293. **Problem.**—*To find the limit to the mechanical advantage on an inclined plane, the co-efficient of friction being given.*

Let α be the limiting angle sought, and let $(\alpha + d\alpha)$ be another angle very near this, so that $d\alpha$ is exceedingly small. Then the equations for the values of P in these two cases are,

$$P = W \sin \alpha + \mu W \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$P = W \sin (\alpha + d\alpha) + \mu W \cos (\alpha + d\alpha)$$

$$= (W \sin \alpha + \mu W \cos \alpha) \cos d\alpha + (W \cos \alpha - \mu W \sin \alpha) \sin d\alpha \quad (2.)$$

Since $d\alpha$ is very small, we may put

$$\sin d\,a = d\,a,\ \cos d\,a = 1 - \frac{(d\,a)^2}{2}.$$

Then the equation (2) becomes

$$P = W \sin a + \mu\, W \cos a + (W \cos a - \mu\, W \sin a)\, d\,a$$

$$- (W \sin a + \mu\, W \cos a) \frac{(d\,a)^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Now let us suppose that the value of a is such that

$$W \cos a - \mu\, W \sin a = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Then the co-efficient of $d\,a$ vanishes; therefore, in this case, whether $d\,a$ be positive or negative, the value for P with the inclination $(a + d\,a)$ will be *less* than the value for P with the inclination a, by the quantity $(W \sin a + \mu\, W \cos a)\frac{(d\,a)^2}{2}$. In such a case P is said to have a *maximum* value for the inclination given by equation (4), that is, for the angle given by

$$\cot a = \mu.$$

Hence it follows that there is a certain angle, depending on the value of μ, for which P is greater than it is if we make $a = 90°$, in which case $P = W$. For this angle, and for angles on either side of it, mechanical advantage is not gained, but on the contrary is actually lost. Since μ is generally a small fraction, this limiting angle is usually not very far from 90°. An inclined plane at all approaching this limiting angle will have a disadvantageous effect. To find the greatest angle at which the plane will have an advantageous effect, we must put $P = W$ in equation (1). Then we have

$$W = W \sin a + \mu\, W \cos a,$$

or

$$\frac{1 - \sin a}{\cos a} = \mu.$$

If a be found from this equation, we shall know that an inclined *plane will be of* no practical use unless it be inclined at a less angle

than a. On the other hand, the equation shows that if it be inclined at a less angle, there is no other maximum or minimum value of P, and therefore the mechanical advantage will always go on increasing the more we diminish the gradient.

§ 12. The Wedge.

294. The wedge is a double inclined plane, movable instead of fixed, as are the planes considered above, and used for separating bodies. The force is applied in a direction perpendicular to the height of the plane, or parallel to the base, and the resistance to be overcome consists of friction and a reaction due to the molecular attractions of those particles of the body which are being separated. This latter reaction will act in a direction at right angles to the inclined surface of the wedge, or length of the plane.

295. **Problem.**—*To find the mechanical advantage in the case of the wedge.*

Suppose the wedge has been driven into the material a distance equal to D C by a force P acting in the direction D C, then it is clear that the energy exerted by P is $P \times DC$. Draw D E, D F perpendicular to A C, B C. Then, since the points E and F of the material were originally together, the work done against the resistance R is $R \times DE + R \times DF = 2R \times DE$.

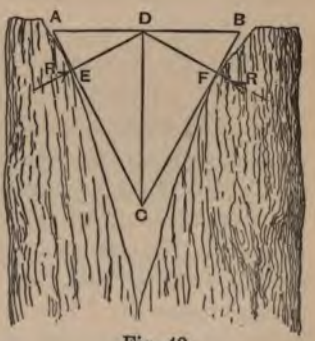

Fig. 49.

Lastly, the friction at E will evidently be represented by μ R, and its point of application will have moved from C to E. Hence the work done upon it is $\mu R \times EC$. Therefore the equation of Energy, or of Virtual Velocities, will be in this case

$$P \times DC = 2R \times DE \times 2\mu R \times EC.$$

If the angle of the wedge, or A C B, be 2α, this becomes

$$P = 2R \times \sin\alpha + 2\mu R \times \cos\alpha.$$

296. **Cor.** If friction be neglected, the equation is

$$P = 2R \times \sin\alpha.$$

It follows that the smaller the angle of the wedge, the greater will be the mechanical advantage gained.

297. In general, the wedge is driven, not by a steady pressure, but by a series of violent blows. Now between the blows there is a powerful reaction R, acting to push the wedge back again out of the cleft, and this is resisted by the friction, and in some cases by the weight of the wedge. The latter is small, and may be neglected. Resolving the two reactions and the two resistances of friction, all parallel to C D, the limiting equation of equilibrium will be

$$2R \sin\alpha - 2\mu R \cos\alpha = 0.$$

The value of α thus obtained is $\alpha = \tan^{-1}\mu$. If α be greater than this value, the wedge will fly back again after being struck, as sometimes happens in practice.

§ 13. The Screw.

298. The screw is another application of the principle of the inclined plane, in cases where a straight inclined plane would be impracticable. Thus, suppose it is wished to provide access to the top of a lofty round tower, at a slope up which people can walk conveniently. To make a long straight staircase, or inclined plane, reaching from the bottom to the top, would be very expensive, and perhaps impossible. But if the inclined plane be made to wind round and round the inside of the tower, still keeping the same angle, the ascent may be constructed with ease. This forms the ordinary spiral staircase. If the tower be large, so that the staircase projects but a little way from the wall, and if the steps be replaced by an inclined plane, we have a representation on a large scale of what is called a female screw, or an ordinary nut. The projecting *rib or staircase* is called the thread, the distance by which it rises

vertically in one complete turn is called the pitch, and its slope to the horizon is called the angle of the screw. The bolt, or male screw, is a cylinder, having an exactly corresponding thread cut on its *outside*, and just of a size to fit into the nut. The work is done by holding the bolt fast, and turning the nut round (or *vice versâ*), when the nut screws itself along the bolt, this being the only way in which motion is possible.

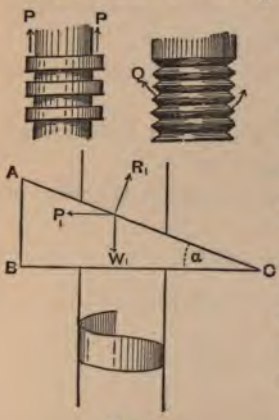

Fig. 50.

299. There are two modes of forming the thread in practice, which are indicated in Fig. 50; in which it is also shown how one complete turn of the thread may be represented by the wrapping of a triangle round the cylinder. The reactions of the nut will, of course, be perpendicular to the surface of the thread at each point, as P and Q; but if we consider any particular point on the thread, we shall see that the forces acting on it are as follows:—

(1.) The resistance to motion W_1, acting parallel to the axis of the screw;

(2.) The reaction R_1 at right angles to the thread;

(3.) The frictional resistance, μR_1, parallel to the thread;

(4.) A force, P_1, produced by the power; this we may consider as acting at right angles to W, and tending simply to turn the screw on its axis.

If α be the angle of the thread, we have, by resolving the forces perpendicular to the thread,

$$R_1 = W_1 \cos \alpha + P_1 \sin \alpha,$$

which gives the value of the reaction R_1

If we form the equations of equilibrium for this and every other point, and add them together, we shall obtain the general equations for the screw: but this would be a troublesome process. The principle of virtual velocities saves us this trouble.

300. **Problem.**—*To find the relation between the power and the weight in the screw.*

Suppose the power P, which need only be employed to make the screw revolve, to act in a plane at right angles to the axis, and at the end of an arm r. Now, suppose the screw to have made one complete turn, and consider what the displacements will be for the several forces.

(1.) The power P will have been moved through the whole circumference of its circle of motion, or through $2\pi r$;

(2.) The weight W will have been moved through the distance A B, or the pitch;

(3.) The total friction, μ (W cos α + P sin α), will have moved through the whole length of the thread, or A C.

The reaction R will not appear, because the motion is always at right angles to its direction.

Hence the equation of virtual velocities will be

$$P \times 2\pi r = W \times AB + \mu\,(W \cos\alpha + P \sin\alpha) \times AC.$$

Since $AC = \dfrac{AB}{\sin\alpha}$, this equation may be written

$$P \times (2\pi r - \mu \times AB) = W \times AB\,(1 + \mu \cot\alpha).$$

301. **Cor.** If friction be neglected, the equation becomes

$$P \times 2\pi r = W \times AB.$$

302. It will be seen that the mechanical advantage gained is greater as the pitch A B is less. In some cases, as in ordinary bolts and nuts, it is desirable that the screw should work at fair speed, and then the pitch must not be too small. In cases where the screw is specially used for obtaining pressure (as in screw-presses for cotton, &c.), this does not apply; but a practical limit to the pitch is given by the fact that, if the angle of inclination is very flat, the threads run so near each other as to be too weak. In that case the screw is apt to "strip its thread," that is, to tear bodily out of the hole, leaving the thread behind.

303. Where great pressure is required, it may be obtained by *means of the* Hunter screw, in which the main screw has another

screw, of somewhat finer pitch, fitting into it, and turning in the opposite direction. The advance of the screw at one turn is then only the difference between the two pitches, and this difference may be made as small as we like without weakening the thread of either.

PART III.—KINEMATICS.

§ 1. Motion in one Straight Line.

304. Kinematics, taken as a separate branch of Mechanics, deals with problems of motion only, without reference to the forces which cause the motions.

305. A body, left to itself, moves uniformly in one straight line, and any change in this motion must be due to some cause, that is to say, to some force. It follows, therefore, that in every problem of motion the existence of some force must be assumed, and therefore that everything in Kinematics falls really under the head of Dynamics. Strictly speaking, this is true; but there are certain cases in which, the forces being simple and constant, the changes of motion which they produce may be studied by themselves, and as if they existed of themselves, without any further reference to the forces which cause them. These are the cases which will be studied in the present division. They may practically be classed under one heading—namely, as those in which there is only one force, and that constant in direction and amount. In this case the acceleration is also constant, and we may treat it as a change in the velocity, independently of the force which causes it.

306. The ordinary example of such a case is that of a body moving under the action of gravity, which for small distances, vertical or horizontal, may be treated as constant in direction and amount. In practice other forces, especially the resistance of the air, act on a flying body; these for simplification are omitted, although in practical cases at high speeds (*e.g.*, in artillery) the resistance of the air must be taken into account.

307. We shall treat this subject, therefore, as applying to bodies *moving in vacuo* under the action of gravity—that is, with a

constant acceleration g, of about 32·2 feet per second ; but it must be remembered that they apply equally to bodies under any other constant acceleration, f.

308. The subject falls into two divisions, according as the initial velocity of the body is, or is not, in the same direction as the acceleration. We shall begin with the first division. In this it is evident that the whole of the motion under gravity will be in one vertical line. We shall assume that velocities are positive when measured upwards.

309. **Problem.**—*To find the equations of motion for a body* in vacuo *moving in a vertical line under the action of gravity.*

These equations are precisely those which have been deduced in Part I., Art. 90. If u be the initial velocity of the body at time $t=o$, v its velocity at the time t, and s the space it has described, then

$$v = u - g\,t$$

$$s = t\,u - \frac{g\,t^2}{2}$$

$$v^2 = u^2 - 2\,g\,s.$$

Here u must, of course, be taken positive or negative, according as the initial velocity is upwards or downwards.

These formulæ will enable us to solve any problems with regard to bodies moving in one vertical line.

310. **Problem.**—*To find the time during which a body rises when projected vertically upwards.*

Let u be the velocity of projection. Then

$$v = u - g\,t$$

is the velocity of the body after any time t. But when the body has reached its highest point, its velocity is zero. If, therefore, we put $v=o$ in the above equation, the corresponding value of t will be the time of rising.

$$\therefore\ t = \frac{u}{g}.$$

311. **Problem.**—*To find the time of flight*, i.e., *the whole time before returning to the starting point.*

The formula $s = ut - \frac{gt^2}{2}$ gives the distance of the body from the starting-point after t seconds. Now, when the body, having risen to its maximum height, has returned to the point of projection, $s = 0$. If, therefore, we put $s = 0$ in the above equation, the value of it will be the time of flight; but from the equation

$$ut - \frac{gt^2}{2} = 0$$

we get $t = 0$, or $t = \frac{2u}{g}$. The former of these two values corresponds to the instant when the body starts; the latter to the instant when the body has returned. Hence $\frac{2u}{g}$ is the whole time of flight.

312. **Cor. 1.**—Since $\frac{u}{g}$ has been proved (Art. 310) to be the time of rising, $\frac{u}{g}$ must also be the time of falling; *i.e., the time of rising equals the time of falling.*

313. **Cor. 2.**—The final velocity $= g \times \frac{u}{g} = u$; hence a body returns to any point in its path with the same velocity at which it left it.

314. **Problem.**—*To find the height to which the body will rise.*

Generally $v^2 = u^2 - 2gs$; but $v = 0$ at the summit, and the corresponding value of s will give the height to which the body will rise;

$$\therefore \quad u^2 = 2gs,$$

or

$$s = \frac{u^2}{2g}.$$

315. **Cor.**—Since $u^2 = 2gs$, where s is the height from which a body falls to gain the velocity u, it follows that a body will

rise through the same space in losing a velocity u as it would fall through to gain it.

316. By an extension of the same fundamental formulæ, we may treat the case of a body sliding from rest down a smooth inclined plane.

317. In this case we must conceive the force of gravity to be resolved into two parts, one parallel to the plane, the other perpendicular to it. The latter will merely produce pressure on the plane, while the former will accelerate the motion of the body, and is the only part with which we shall be here concerned.

318. **Problem.**—*To find the acceleration with which a heavy body will move, if it slides down or is projected up a given inclined plane.*

Let the body be moving down the plane A B, and suppose it to be at a.

Draw $a\,b$ vertically downwards, and make it equal to A B. Take $a\,b$ to represent g, the vertical acceleration of gravity. Then, if $b\,c$ be drawn at right angles to A B, the triangle $a\,b\,c$ is in every respect equal to the triangle A B C, and $a\,c$ represents the resolved part of the acceleration g in the direction A B. Hence, if f be the required acceleration,

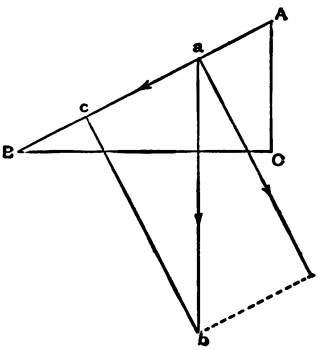

Fig. 51.

$$f = \frac{a\,c}{a\,b} \times g = \frac{\text{AC}}{\text{AB}} \times g.$$

Let A C, the height of the plane, equal h, and A B, the length of the plane, equal l.

Then

$$f = \frac{h}{l} \times g.$$

If the body be projected up the plane, the retardation due to the

body's tendency to fall will also be represented by ac, and will be equal to $\frac{h}{l} \times g$.

The motion of the body, if allowed to fall down the plane, can be ascertained by putting $f = \frac{h}{l} g$ (or $f = g \sin \alpha$) in the equations

$$v = ft\,; \quad s = \frac{ft^2}{2}\,; \quad v^2 = 2fs.$$

If the body be *projected* up or down the plane, the motion can be determined by substituting this value of f in the equations

$$v = u \pm ft\,; \qquad s = ut \pm \frac{ft^2}{2}\,; \qquad v^2 = u^2 \pm 2fs.$$

319. **Problem.**—*To find the velocity acquired by a body in falling down a given inclined plane.*

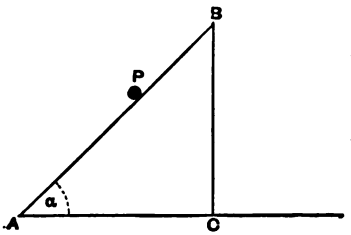

Fig. 52.

Let A B be the inclined plane, α its inclination, P the place of the body at a given time,

B P $= s$, $v =$ the velocity at P;

then we have by our formula, since $u = o$,

$$v^2 = 2g \sin \alpha \times s,$$

which gives the velocity. If we draw B C perpendicular to the horizontal line through A, and call V the velocity at A, we have

$$V^2 = 2g \,.\, \text{A B} \sin \alpha$$

$$= 2g \,.\, \text{B C}.$$

Hence the velocity is the same at A, as if the body had fallen through the vertical space B C: that is to say, the velocity generated by gravity depends solely upon the vertical space through which it is allowed to act; a result which might, perhaps, have been anticipated.

320. *Problem.—A body is projected upwards along an inclined*

plane with a given velocity; to find how high it will ascend, and the time of ascent.

If v be the velocity at the time t, when the body has ascended through a space s, α the angle of the plane, and V the given velocity of projection, we have

$$v^2 = V^2 - 2gs \sin \alpha.$$

When $v = 0$ the body will stop; hence, the distance required will be given by the equation

$$s = \frac{V^2}{2g \sin \alpha}.$$

For the time, we have the equation

$$v = V - gt \sin \alpha;$$

and the body stops when $v = 0$, or when

$$t = \frac{V}{g \sin \alpha}.$$

321. **Theorem.**—*Let A C B be a circle in a vertical plane, A its highest point; the time of descent down all chords drawn through A, considered as inclined planes, will be the same.*

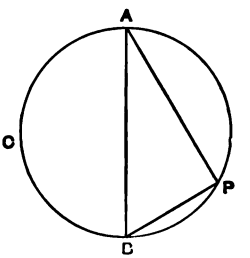

Fig. 53.

Let A P be any chord, α its inclination to the horizon: draw the vertical diameter A B, and join B P, then $A B P = 90° - B A P = \alpha$. Now if t be the time of descent down A P, we shall have (Art. 318)

$$A P = g \sin \alpha \frac{t^2}{2};$$

but, $A P = AB \sin \alpha$,

$$\therefore AB = \frac{gt^2}{2},$$

and $$t = \sqrt{\frac{2AB}{g}}.$$

This value of t is independent of α, and is therefore the same for all chords.

322. **Problem.**—*From a given point to draw the line, down which as an inclined plane a particle will descend to a fixed line in the shortest time possible.*

Let A be the given point, and through it draw a horizontal line to meet the given line in B; in the given line take BC equal to BA; join AC, which shall be the line required.

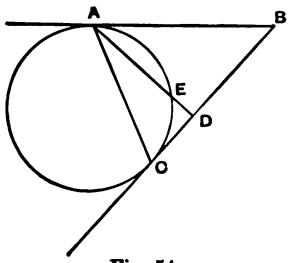

Fig. 54.

It is easily seen by geometry that a vertical circle having A for its highest point will *touch* the given straight line in C. Now if AC be not the line of quickest descent, let AED, cutting the circle just mentioned in E, be the required line; then the time of descent down AC is equal to the time down AE, and therefore is *less* than the time down AD. Therefore AC is the line of quickest descent.

323. Similarly it may be shown, that in general the line of quickest descent from a point to any given curve will be found by describing a circle, to touch the horizontal line passing through the given point at that point, and also to touch the given curve. The chord joining the points of contact will be the line of quickest descent.

§ 2. Motion of Projectiles.

324. We now go on to the case where the initial velocity of the body is not vertical, which is that of a body flying through space, such as a cricket ball or a rifle bullet; in this case the body will clearly describe a curve, and our first object is to discover the nature of this curve.

325. **Def.**—*A body flying through space under the action of gravity is called a Projectile; and the path which it describes is called its Trajectory.*

326. **Theorem.**—*The path of a projectile* in vacuo *will be a parabola.*

It is evident that the path will lie in the vertical plane through the direction of projection: let it be the plane of the paper, and let P (Fig. 55) be the point of projection, P N R the line in which the body is projected, which will manifestly be a tangent to the curve described.

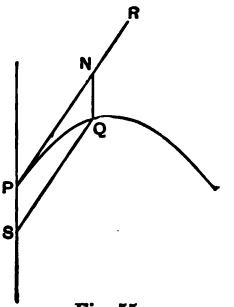

Fig. 55.

Let V be the velocity of projection, and N the point at which the body would arrive with this velocity in the time t, so that

$$P N = V t.$$

From N draw N Q vertical, and make $N Q = \frac{g t^2}{2}$; then, since the space which the body would describe in the time t, under the action of gravity only, is N Q, and since if gravity had not acted the body would have been at N, therefore when the body is simultaneously under the action of its original velocity V, and that generated by gravity, it will be at Q.

Complete the parallelogram P S Q N; then

$$P S = N Q = \frac{g t^2}{2},$$

$$Q S = P N = V t;$$

$$\therefore Q S^2 = V^2 t^2 = \frac{2 V^2}{g} \times P S = 4 P S \times \frac{V^2}{2g}.$$

But in the parabola $Q S^2 = 4 P S \times O P$, where O is the focus. Hence Q lies in a parabola, of which the axis is vertical, and the distance of P from the directrix or focus is $\frac{V^2}{2g}$.

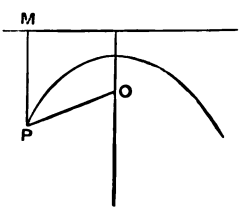

Fig. 56.

Cor.—From the point P (Fig. 56) draw P M perpendicular to the directrix; then

$$V^2 = 2g \,.\, O P = 2g \,.\, P M.$$

Hence, *the velocity at any point of the parabola is that which would be acquired in falling from the directrix.*

327. **Problem.**—*To determine the position of the focus of the parabola.*

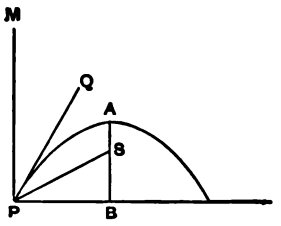

Fig. 57.

Let P be the point of projection, P Q the direction of projection, A S B the axis of the parabola, P M vertical, P B horizontal; and let Q P B (which is called the angle of projection) $= \alpha$.

From P draw P S, making the same angle with P Q as P Q makes with P M; then, by a property of the parabola, the point S, in which P S meets the axis, is the focus.

$$\text{Now } SPB = QPB - QPS = QPB - MPQ$$

$$= \alpha - (90° - \alpha) = 2\alpha - 90°;$$

$$\therefore PB = SP \sin 2\alpha, \text{ and } SB = -SP \cos 2\alpha;$$

$$\text{but } SP = \frac{V^2}{2g}; \quad \therefore PB = \frac{V^2}{2g} \sin 2\alpha, \text{ and } SB = -\frac{V^2}{2g} \cos 2\alpha.*$$

The lines P B, S B determine the position of S; for we have only to measure the distance P B horizontally and B S vertically, and we shall determine S. Or we may determine the position of the line P S as above, and take in it a point the distance of which from P is $\frac{V^2}{2g}$; this will be the focus.

328. Cor.—If $\alpha = 45°$, the focus of the parabola is in the horizontal plane passing through the point of projection.

329. **Problem.**—*To determine the greatest height to which the projectile will rise.*

*The quantity $-\frac{V^2}{2g} \cos 2\alpha$ will be *positive*, because in the figure 2α has been supposed to be greater than 90°.

The velocity of projection may be supposed to be resolved into two velocities, one horizontal = V cos α, the other vertical = V sin α. The vertical motion of the body will be the same as if it were projected vertically with this latter velocity; hence, if h be the height required, we have

$$h = \frac{V^2 \sin^2 \alpha}{2g}.$$

330. **Problem.**—*To find the* latus rectum *of the parabola.*

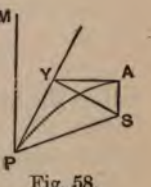

Fig. 58.

Draw the perpendicular S Y on the tangent P Y, which is the line of projection; let A be the vertex, then A Y is a tangent at the vertex, and therefore horizontal.

Then

$$S\,Y\,A = 90° - \alpha, \text{ and } S\,P\,Y = M\,P\,Y = 90° - \alpha;$$

$$\therefore \text{ the } \textit{latus rectum} = 4\,A\,S = 4\,S\,Y \cos \alpha = 4\,S\,P \cos^2 \alpha$$

$$= \frac{2\,V^2}{g} \cos^2 \alpha.$$

331. **Problem.**—*To find the Range of the projectile, that is, the distance from the point of projection at which the body meets the horizontal plane through the point of projection; and also the time of flight.*

The range will evidently be twice the distance of the point of projection from the axis of the parabola. But this distance was shown in Art. 327 to be $\frac{V^2}{2g} \sin 2\alpha$.

$$\therefore \text{ the range} = \frac{V^2}{g} \sin 2\alpha.$$

To find the time of flight we have only to observe, that the horizontal velocity of the body is uniform and equal to V cos α, and therefore that the time which elapses between the moment of projection and the moment of the body striking the ground is that occupied by a body moving through the space $\frac{V^2}{g} \sin 2\alpha$ with the velocity V cos α.

$$\therefore \text{ the time of flight} = \frac{V^2 \sin 2\alpha}{g V \cos \alpha} = \frac{2V}{g} \sin \alpha.$$

The same result may be arrived at from the consideration of the vertical motion of the body. For since the vertical velocity of projection is V sin α, the time of flight will be that which elapses before a body projected vertically with a velocity V sin α falls and strikes the ground; and this time is manifestly expressed by $\frac{2V \sin \alpha}{g}$, since the time of ascent will be $\frac{V \sin \alpha}{g}$ and the time of falling the same.

332. **Cor.**—The greatest value which $\sin 2\alpha$ can have is 1, which corresponds to $\alpha = 45°$; hence, the range of a projectile will be greatest when the angle of projection is 45°.

333. **Problem.**—*To find the relation between the velocity and angle of projection, in order that the projectile may strike a given point.*

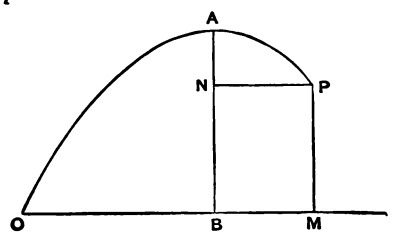

Fig. 59.

Let O be the point of projection, O M horizontal, and M P vertical; let P be the given point, and let O M = h, M P = k. Draw A B the axis of the parabola, and P N perpendicular to A B.

Then we have proved (Arts. 327, 329) that

$$O B = \frac{V^2 \sin 2\alpha}{2g},$$

$$\text{and } A B = \frac{V^2 \sin^2 \alpha}{2g};$$

$$\therefore A N = \frac{V^2 \sin^2 \alpha}{2g} - k, \qquad P N = h - \frac{V^2 \sin 2\alpha}{2g}.$$

Also the latus rectum $= \frac{2V^2}{g} \cos^2 \alpha$; hence

$$\mathrm{P\,N^2} = \frac{2\,\mathrm{V}^2}{g}\cos^2\alpha\,.\,\mathrm{A\,N};$$

$$\text{or } \left(h - \frac{\mathrm{V}^2 \sin 2\alpha}{2g}\right)^2 = \frac{2\,\mathrm{V}^2}{g}\cos^2\alpha\left(\frac{\mathrm{V}^2\sin^2\alpha}{2g} - k\right);$$

$$\text{or } h^2 - 2h\frac{\mathrm{V}^2}{g}\sin\alpha\cos\alpha + 2k\frac{\mathrm{V}^2}{g}\cos^2\alpha = 0,$$

which is the relation between α and V required.

§ 3. General Theorems.

334. There are one or two general facts with respect to the motion of bodies, which are properly included under the head of Kinematics. The first of these is the conception of Angular Velocity, as distinguished from Linear Velocity.

Definition.—*If the motion of a particle in a plane be considered with reference to a fixed point in that plane, the angular velocity of the former about the latter means the rate of increase of the angle, which the line joining the two makes with some fixed line in the plane.*

335. By the rate of increase of the angle, we mean the ratio of any given increase of the angle (supposed to take place uniformly) to the time during which it takes place. If the increase of the angle be $d\theta$, and the time dt, and if both be very small, then we may always consider the increase to be uniform: hence we shall have, as the proper value of the angular velocity ω,

$$\omega = \frac{d\theta}{dt}$$

The increase of the angle, in angular velocity, thus takes the place of the increase of the distance from a fixed point in linear velocity.

336. **Cor.**—If the increase of the angle be constant for any time t, we shall have

$$\omega t = \theta.$$

337. The following general proposition as to the relative move-

ments of the particles in a system is a very important one, from the convenience it affords in solving dynamical problems. It will be observed that no condition whatever is attached to the motion, and therefore the proposition holds, whatever be the forces acting on the system.

338. Theorem.—*If we suppose the velocity of any one particle of a system, reversed in direction, to be communicated to each particle of the system in addition to that which it already possesses, then the relative motions of all about the first thus reduced to rest, will be the same as their relative motions about it when all were in motion.*

It is evident that if at every instant the distance of two particles, and the direction of the line joining them, be the same as for two other particles, the relative motions of one of each pair about the other will be the same.

Now let P and Q be the positions which the first particle and any of the others would have occupied (if undisturbed) at the end of a time δt. Suppose that the same additional velocity v, in any direction Pp, is impressed on both during the interval δt; then the particles will be at the end of the interval in the positions p, q, and pq is evidently equal and parallel to PQ. This will be true for all indefinitely small values of δt, and therefore at every instant the *relative* motion of the two particles p, q is the same as that of P, Q. But by impressing on the whole system a velocity equal and opposite to that of P, the latter is reduced to rest. Whence the proposition.

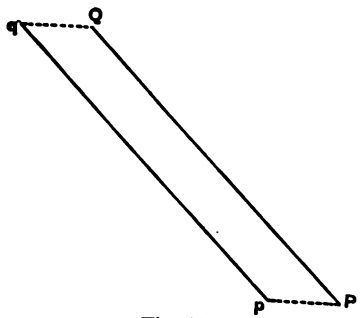

Fig. 60.

339. We have given a formal proof of the above proposition, but in fact it follows immediately from the principle that all known motion is relative. Thus the motion of P, which we suppose to be known, could not have been absolute motion, but merely motion relative to some point O assumed to be fixed, such as the centre of

the earth. Therefore we may, if we choose, take P to be the fixed point, instead of O; and this we shall clearly do if we suppose each particle to have impressed upon it a velocity equal and opposite to the velocity of P relative to the point O, originally supposed to be fixed.

340. We are thus enabled to determine, from the actual motions of all the particles of a system at any time, their relative motions with respect to any one of them; or conversely, from the actual motion of one particle, and the motions of the others relative to it, we can determine the actual motions of the latter. In practice, of course, the term "actual motion" can only mean motion relative to some point generally taken as fixed. In many examples of dynamics, the action of the forces becomes much simpler and easier to grasp, if we can take some particular point in the system as fixed; and this is always possible, on the above principle, by finding the velocity of this point, and impressing that velocity, but in the reverse direction, on every particle of the system.

341. Two examples of the above principles may be given.

342. **Theorem.**—*Two particles will always describe similar orbits about each other, and about their common centre of gravity.*

Let A and B be the particles, G their centre of gravity at any time. Then $\frac{AG}{GB}=\frac{B}{A}$, if A and B represent their masses, and this is a constant ratio.

The path of B about A will evidently be the same as that of A about B, since the length and direction of the line AB are the same whichever end be supposed fixed. Also if G be fixed, the path of B about it will evidently differ from that of B about A by having corresponding *radii vectores* diminished in the ratio $\frac{A}{A+B}$. But this is the definition of similar curves. The same, of course, would hold with respect to the relative path of A with respect to G. This proposition enables us materially to simplify the equations of motion of two mutually attracting free particles.

343. **Problem.**—*Suppose two particles, one of which* moves

uniformly in a straight line, while the other moves uniformly in a circle about it as centre; to determine the path of the latter, the motion being in one plane.

Take the line of motion of the first as the axis of x, and let v be its velocity; let the plane of the circle be the plane of xy, a the radius of the relative circular orbit, ω the angular velocity in it. Suppose the revolving particle to be initially in the axis of x. Also at time t suppose the line joining the particles to be inclined at an angle θ to the axis of x. Then for the co-ordinates of the revolving particle we have

$$y = a \sin \theta,$$

$$x = vt + a \cos \theta.$$

But

$$\theta = \omega t\ ;$$

$$x = vt + a\sqrt{1 - \frac{y^2}{a^2}} = \frac{v}{\omega}\sin^{-1}\frac{y}{a} + \sqrt{(a^2 - y^2)}\ ;$$

which is the equation to the absolute path required.

PART IV.—DYNAMICS.

§ 1. Introductory.

344. The principles on which the Science of Dynamics proper is founded have been already laid down in Part I. We now go on to consider how these principles may be used, as in the case of Statics and Kinematics, to develop the leading facts relating to this branch of our subject.

345. The central problem in Dynamics may be stated as follows:—*Given the mass, velocity, and position of every particle of a system at a given time, and also the forces acting upon it, to determine their velocities and positions at any subsequent time.*

This problem is immeasurably more difficult than the corresponding central problem in Statics (Art. 202), and we shall not attempt its solution. The reason of this difficulty is that in this case the relative positions of the particles are continually altering, and this alteration causes variations in the forces which they exercise upon each other (by the definition of matter). All these variations have to be taken account of, and they complicate the question so much that even in the case of three particles, free to move under their mutual attractions, the problem, in the form stated above, has never been solved. This problem, which is known as that of the Three Bodies, is the fundamental problem of the Lunar theory; but it is there solved only by approximate methods, based on the circumstance that one of the three bodies —the sun—is very much larger than either the earth or the moon.

346. Hence, to study Dynamics at all, it is necessary to make certain assumptions for the purpose of simplifying the question. Of general assumptions, the most convenient is that the system is a *rigid* one—*i.e.*, that the geometrical relations of its particles, or their positions relative to each other, are always constant (Art. 176). This, of course, never holds exactly in nature, but it holds approxi-

mately in cases where the external forces are small in comparison with the internal forces, which tend to resist any change in the geometrical relations of the system. This branch of Dynamics is called Rigid Dynamics: but even this, if treated in a general manner, is of great difficulty, and requires the aid of high mathematics.

All that we can do, therefore, in Dynamics is to investigate certain cases, in which by special assumptions very great simplicity may be attained. In some of these cases the hypothesis of rigidity will be introduced, in others not.

347. Generally, it may be said that the aim, in all dynamical problems, is to find the velocity and the resultant acceleration of each particle of the system, at a given instant. When these are known, its path, for the next instant, is given at once by the ordinary equations of acceleration (Art. 90).

§ 2. Motion of Two Bodies under Gravity only.

348. In this class of problems the acceleration is constant, and it might be supposed that they should be treated, like those of projectiles, in the chapter on Kinematics. But in these problems it will appear that the acceleration can only be determined when the mass moved has been considered. We have therefore to use, not only the equations of acceleration, but the fundamental equation connecting the statical and dynamical measures of force. This equation is (Art. 125)

$$P = Mf,$$

where P is the resultant force or unbalanced effort (Art. 142), acting on the system, measured in the pounds weight it will lift;

„ f is the acceleration, or change of velocity produced on the system, measured in feet per second;

„ M is the mass moved, the unit of mass being the mass of g lbs., where g is the accelerating force of gravity, and $= 32{\cdot}2$ nearly.

349. **Problem.**—*To find the motion of two bodies, the masses of which are M and M′ respectively, and which are connected by a fine*

weightless string passing over a smooth peg or small pulley, the only effect of the peg being to change the direction of the string.

Let M be greater than M′; then the moving force producing motion, or the unbalanced effort (Art. 142), is the difference of the weights, or $Mg - M'g$; but the whole mass moved is $M + M'$; consequently (by the equation $P = Mf$) the accelerating force is $\frac{M - M'}{M + M'}g$.

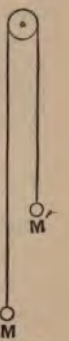

Fig. 61.

Hence, if x be the distance of M from the peg at the time t, v its velocity, a the distance when the motion commenced, we shall have

$$v = \frac{M - M'}{M + M'}g\,t$$

$$x = a + \frac{M - M'}{M + M'}\frac{g\,t^2}{2}.$$

350. **Problem.**—*In the same case, to determine the tension of the string.*

To do this we observe, that if the weight Mg were suspended at the extremity of a string and were at rest, the tension would be Mg, the force being entirely employed in producing tension. As it is, however, a portion of the force acting on M is employed in producing motion, and the remainder in producing tension of the string. Now, the acceleration corresponding to the former portion we have determined to be $\frac{M - M'}{M + M'}g$, and the acceleration corresponding to the whole is g. Hence the acceleration f corresponding to the tension T is $(g - \frac{M - M'}{M + M'}g)$, or (since $T = Mf$)

$$\text{the tension } T = M\left\{g - \frac{M - M'}{M + M'}g\right\}$$

$$= \frac{2\,M\,M'}{M + M'}g\,;$$

an expression which, it may be observed, involves M and M′ symmetrically, as manifestly it ought.

351. This problem may be solved in another manner, which will give us at once the tension of the string and the accelerating force.

Thus, let T be the tension of the string; then the moving force upon the body M will be $Mg - T$, and the accelerating force therefore $g - \frac{T}{M}$. In like manner, the accelerating force upon M' will be $g - \frac{T}{M'}$; and these two must be equal, but of opposite algebraical signs, since one of the bodies necessarily *ascends* with the same velocity with which the other *descends*.

Therefore,

$$g - \frac{T}{M} = -g + \frac{T}{M'},$$

or

$$T\left(\frac{1}{M} + \frac{1}{M'}\right) = 2g,$$

$$T = \frac{2MM'}{M+M'}g, \text{ as before.}$$

And the accelerating force $= g - \frac{T}{M} = g - \frac{2M'}{M+M'}g = \frac{M-M'}{M+M'}g.$

EXAMPLE.—Suppose $M = 2M'$, and $a = o$, to find how far M will descend in 1". We have

$$x = \frac{2-1}{2+1}\frac{g}{2} = \frac{16.1}{3} = 5{\cdot}3 \text{ feet.}$$

Also in this case the tension of the string $= \frac{2}{3} Mg =$ two-thirds of the heavier weight.

352. **Problem.**—*Two weights are placed upon two opposite inclined planes, and connected by a fine string which passes over a smooth pulley at the highest point of the planes: to determine the motion.*

Let A B, A C be the two planes, α, β their respective inclinations.

Then the part of the weight Mg which is effective in producing motion is $Mg \sin \alpha$, and that of $M'g$ is $M'g \sin \beta$; the difference, or $Mg \sin \alpha - M'g \sin \beta$ is the moving force; the mass moved is $M + M'$; therefore the accelerating force is

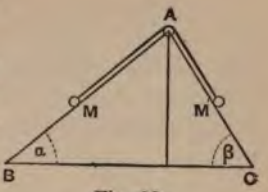

Fig. 62.

$$\frac{M \sin \alpha - M' \sin \beta}{M + M'} g.$$

If x be the distance of M from A at the time t, a the distance at the beginning of the motion,

$$x = a + \frac{M \sin \alpha - M' \sin \beta}{M + M'} \frac{gt^2}{2}.$$

The tension of the string may be found as in Art. 350.

EXAMPLE.—Suppose $M = 3M'$, $\alpha = 30°$, $\beta = 60°$: then,

$$x = a + \frac{\frac{3}{2} - \frac{\sqrt{3}}{2}}{3+1} \times \frac{gt^2}{2}$$

$$= a + \frac{3 - 1.7}{16} gt^2 = a + 2.6\, t^2, \text{ nearly.}$$

353. The case of two falling bodies connected by a string is especially interesting, because it enables us to prove by direct experiment the constancy of gravity, and its amount. A weight suffered to fall freely acquires almost instantaneously so high a velocity that accurate measurement is difficult; and, moreover, the resistance of the air at once begins to have a considerable effect. But in the case of Art. 349, it will be seen that the velocity at any moment is proportional to $M - M'$, and therefore, by making these very nearly equal, we are able to reduce the velocity as much as we please.

Its measurement then becomes easy, and the resistance of the air becomes at the same time insignificant. On the other hand, some fresh conditions are imported into the problem, as the weight and stiffness of the string, and the friction upon the pulley; but by various contrivances these may be got rid of or allowed for. An apparatus arranged for this purpose is called an Atwood's machine, from the name of its inventor, and is often seen in physical laboratories.

§ 3. Impulsive Forces.

354. Forces in dynamics are usually divided into continuous forces and impulsive forces; and though there is no essential difference between the two, yet they require different treatment, and must therefore be separately considered.

355. A continuous force is such as we have already discussed in Part I., acting continuously for a finite time, and producing a finite change of velocity in that time. Such a force is measured, in its effect on a body of mass M, by the product of the mass it moves, and the acceleration, f, which it causes—that is, by the increase which it causes in M's momentum during a unit of time (Art. 68): and the momentum which it generates in any other time t is expressed by $\mathrm{M}ft$. Again, its total effect on the mass, while acting upon it over any space, s, is expressed by the energy exerted, $\mathrm{M}fs$, or $\frac{\mathrm{M}(ft+u)^2-\mathrm{M}u^2}{2}$, where u is the initial velocity in the direction of the force (Art. 129).

356. So long as the acceleration of a force is of moderate amount, compared to the ordinary units of space, and so long as the time during which it acts is not very small, there is no difficulty in applying the above formula. If the acceleration be small, and the time small also, the measurement of the momentum becomes difficult: but as this momentum is then insignificant, this difficulty is of no importance. If, however, the force be very great, and the time very small, the difficulty becomes serious. A typical case is the blow of a hammer. Here the time during which there is *contact is* apparently instantaneous—certainly too small to be

measured by any ordinary methods : yet the effect produced is very considerable. In such cases it is impossible accurately to determine f and t; but we can determine their product, or ft, since this is merely the change in velocity caused by the blow. Hence, in the case of blows, or impulsive forces, we do not attempt to measure the force and the time of action separately, but simply take the whole momentum produced, as the measure of the impulse. Similarly, we cannot measure the space s described under the action of the force, in order to determine the work done; but must content ourselves with measuring the change in the kinetic energy, or in $\frac{Mu^2}{2}$, in which there is generally no difficulty, since the change in the velocity can easily be determined. Hence, we arrive at the following as the definition of an impulsive force.

357. Def.—*An impulsive force is a force so large in amount, and acting for so short a time, that we have to measure its effect simply by means of the change in velocity which it produces.*

358. It was formerly usual to define an impulsive force as an indefinitely large force acting for an indefinitely short time; and then to explain that, while there were, of course, no such forces in nature, there were forces, such as the blow of a hammer, which had to be treated on that supposition. But this seems quite unnecessary. As a matter of fact, improved modes of measurement are frequently enabling us to apply the ordinary formula to forces formerly considered impulsive. Thus the explosion of a cannon would once have been considered almost a type of an impulse : but now that we can measure accurately to the ten-thousandth of an inch, or of a second, the effect of time in an explosion may be studied, and has been studied with important results.

§ 4. Motion of a Particle in a Curved Line.

359. Theorem.—*A particle moving in a curved line, without friction, does not lose any of its velocity in consequence of the reactions which compel it to move in that line.*

Suppose a particle to move along a smooth plane, AB, Fig. 63,

and at B to meet another plane, B C, inclined at an angle α to the former. An impulsive action will then take place, due to the reaction of the plane, B C, which will cause the particle to follow that plane. Since the plane is smooth, this impulse must be normal to the plane B C (Art. 234); and it will be measured by the velocity which it imparts to the particle (Art. 357). Let B M, drawn at right angles to B C, represent this velocity, and let B N represent the original velocity of the particle. Then, by Art. 122, the resultant velocity will be fixed, in amount and direction, by the diagonal of the parallelogram, of which B M, B N are the sides. But we know that this velocity must be in the direction of B C; hence, the fourth point O of this parallelogram must lie on B C. Draw B P perpendicular to O M. Then the velocity B M may be considered as compounded of a velocity P M, which goes to diminish the original velocity B N, and a velocity B P, which is at right angles to the original velocity, and alters the direction of the particle's motion from A B to B C. The values of these two component velocities are:

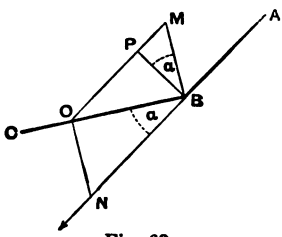

Fig. 63.

$$\text{Velocity } P M = B M \sin \alpha,$$

$$\text{Velocity } B P = B M \cos \alpha;$$

also the resultant velocity, or $B O, = B N \cos \alpha$.

Now let us suppose the angle α to be indefinitely diminished. Then we may consider A B, B C as two successive elements of a smooth curve, considered as made up of an indefinitely large number of indefinitely short planes following each other. But in this case, sin α approaches the value 0, while cos α approaches the value 1. Hence, the ratio of P M to B P, or of the part of the impulse which destroys velocity to the part which changes direction, becomes indefinitely small: or, in the limit, the whole of the impulse is employed in changing direction, and none of it in *destroying velocity*. The same will be true for the sum of any

number of successive impulses; that is, the ratio of the velocity destroyed to the velocity employed in changing direction will be indefinitely small. But the velocity employed in changing direction is itself a finite quantity; therefore the velocity destroyed is an indefinitely small quantity, or the particle does not lose any velocity from being constrained to follow the curve.

The same appears from the expression for the resultant velocity, B N cos α, which, in the limit, becomes equal to B N.

360. The above proposition is often proved by simply saying that the reaction at any point of the curve is normal to the curve, and therefore can have no effect on the velocity, which is tangential to the curve; but since it is clear that, if α is a finite angle, there is a distinct loss of velocity at each change of direction, it seems desirable to show at length that, when α becomes indefinitely small, the sum of all these losses of velocity vanishes.

361. In the proof it has been supposed that the particle is constrained to move in the curve by the resistance of a smooth rigid body, as would be the case with a small ring sliding on a smooth curved wire, or a ball rolling in a smooth bowl. But it is clear that the proof does not depend upon this supposition. The impulse B M may be given in any other way; for instance, by the tension of a string, as in the case of a stone tied to a cord and whirled round in the hand; or by the tension of the inner parts of the same piece, as when a sledge hammer is being swung by a smith; or, lastly, by a force of attraction acting across the intermediate space, as in the case of a planet describing its orbit round the sun. All that is required is that there should be a force of some kind normal to the curve, and thus causing the particle to follow the curve instead of proceeding along the tangent, which, by the first law of motion, it otherwise would do.

362. Any curve may be considered as the limit of a polygon, the number of whose sides is made indefinitely great, and their length indefinitely small. Let A B, B C, Fig. 64, be two successive equal sides, or elements, of such a polygon. Bisect them in D, E, and draw perpendiculars, D O, E O, to meet in O. Then, D O will be equal to E O, and a circle described with radius D O or E O will touch both B C and A B. Hence, A B, B C

may be considered as successive elements of a polygon circumscribed round this circle; and when the sides of the polygon are made indefinitely small, the lines A B, B C will form elements alike of the curve and of the circle. Such a circle is called the circle of curvature for that point of the curve, and O D is called the radius of curvature.

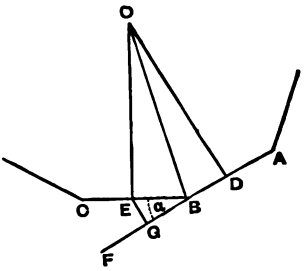

Fig. 64.

363. **Definition.**—*The circle of curvature at any point of a curve is that circle of which two successive elements coincide with two successive elements of the curve; and its radius is called the radius of curvature.*

364. Hence it follows that, to study the conditions of a particle at any point on a curve, we may conceive it to be moving, not in the actual curve, but in the circle of curvature at that point. On this supposition the normal impulse, which keeps the particle on the curve, will pass through the centre of the circle. Such an impulse is called a centripetal impulse (*centrum* = centre, *petere* = to seek). If we now suppose the particle to continue to move in the circle of curvature, instead of following the curve, then, since the circle is a curve everywhere symmetrical, it follows, by the principle of symmetry, that each successive impulse must be equal to the last. The sum of these small equal impulses will take the form of a constant force, always acting towards the centre of the circle; and this force is called the centripetal force. By Art. 359, it also follows that the velocity in the circle will be constant.

365. **Definition.**—*A particle moving uniformly in a circle is under the action of a constant force tending always to the centre; and that force is called the Centripetal Force.*

366. **Problem.**—*To find the centripetal force for a particle of given mass, moving in a given circle with a given velocity.*

Let M be the mass of the particle, V its velocity, and r the *radius O E of the* circle, Fig. 64. Let the angle E B F be α, then

E O D also $= \alpha$, and E O B $= \frac{\alpha}{2}$. Let dt be the element of time during which the particle is describing the space D B E with velocity V. Draw E G perpendicular to B F. Then, if the centripetal force had not acted, the particle would have been, at the end of the time dt, on the line B F, instead of which it is now in the position E. Hence E G represents the space which it has been caused to describe by the centripetal force in the direction normal to its original direction. Hence, by the formulæ for acceleration, we have

$$\text{E G} = \frac{1}{2} f (dt)^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Here f is the acceleration due to the centripetal force, or, in other words, the velocity it will generate in a unit of time.

But E G = E B sin α,

and E B = O E tan E O B $= r \tan \frac{\alpha}{2}$.

Also E B + B D, or 2 E B, is the distance described by the particle, in the time dt, with the velocity V; hence

$$\text{V}\, dt = 2\,\text{E B} = 2\, r \tan \frac{\alpha}{2}.$$

Hence the equation (1) becomes

$$r \tan \frac{\alpha}{2} \sin \alpha = \frac{f}{2} \left(\frac{2\, r \tan \frac{\alpha}{2}}{\text{V}} \right)^2 .$$

But in the limit, when α becomes very small, we may write $\frac{\alpha}{2}$ for tan $\frac{\alpha}{2}$, and α for sin α. Then the equation becomes

$$r \frac{\alpha^2}{2} = \frac{f}{2\,\text{V}^2} \times r^2 \alpha^2$$

whence

$$f = \frac{\text{V}^2}{r}.$$

This is the centripetal acceleration acting on the mass V. The centripetal force is therefore

$$\text{M} \times \frac{\text{V}^2}{r}.$$

367. The above proof does not depend upon the fact that the particle continues to move in the circle, except so far as this is the condition that the centripetal impulses shall all be equal, and therefore the centripetal force constant. Hence, if a particle move in any curve whatever, we may apply the proof to determine the value of the centripetal force *at that point;* that is, the value which the centripetal force would have, if the particle continued to move in the circle of curvature at that point. The centripetal force at that moment is exactly analogous to the velocity at any moment, in the case of a particle moving with variable velocity. The demonstration will be exactly as in Art. 366, taking the circle of curvature for the circle in which the particle moves. Hence we have the theorem: *If a particle of mass* m *move in any curve whatever, it may be taken as acted on at any point by a centripetal force* $= m\frac{V^2}{r}$, *where V is the velocity, and* r *the radius of curvature at that point.*

368. When the moving particle is kept to its curvilinear path by a *tensile* force (as in the case of a stone whirled round at the end of a string), the centripetal force takes the form of a tension in the string. As this must act both ways, the hand holding the string will feel a pull upon it, which, if the circular motion be left out of consideration, may be taken to be the pull due to an actual force, such as would be caused by a weight, equal to $\frac{M V^2}{r}$, and acting at the end of the string. To this supposed force the name Centrifugal Force has been given. But, according to our definition of force, this is not a force at all. For a force is a cause of motion, and if not counteracted, it will always produce motion in its line of action. But suppose, by suddenly cutting the string, we leave this centrifugal force free to operate. Then the stone will not move a single inch in the line of action of this force, but will "fly off at a tangent;" in other words, continue to move in a straight line at right angles to the supposed force. In point of fact, it is not the stone which is pulling at my hand, but my hand which is pulling at the stone, and continually causing it to move in a new direction. If I were pulling it *towards me* along a smooth table, I should in the same way have

to exert force in order to start and increase its motion, but there would be no force upon the stone, tending to make it move away from my hand; and if the string were broken, the stone would continue to move forward with the exact velocity it had at that moment. Centripetal force is expended, not in balancing a centrifugal force, but in causing the body to move in a new direction; and the work done by the force is represented by the actual energy, in that direction, imparted to the body.

369. Centrifugal force is, however, a convenient term to express generally the tendency which bodies have, when whirled round in a circle, to move farther and farther away from the centre. This tendency is very often taken advantage of in the mechanical arts. Thus, in an ordinary centrifugal pump, water is admitted to the centre of a hollow disc, rapidly revolving, and escapes at a high velocity, but of course in a tangential direction, at the circumference. Centrifugal fans, for producing a blast of air, are similarly constructed. Again, since the expression for the so-called force is $\frac{MV^2}{r}$, it will be seen that, other things being the same, the tendency is greater as the mass is greater. Hence, if a mixture of two fluids, one heavier than the other, be set in rapid rotation, the heavier, owing to its greater mass, will pass more rapidly towards the outside, and, if prevented from escaping, will accumulate there. Advantage of this is taken in the separation of cream from milk. The milk being rotated inside a disc at an enormous velocity (several thousand revolutions per minute), the skim milk, which is the heavier, flies to the outside, and issues through a pipe from thence, while the cream is collected nearer the centre. Lastly, the occasional bursting of rapidly revolving articles, such as grindstones, is a special instance of the effect of "centrifugal force;" but of course in such cases the fragments fly off tangentially, not normally, when the fracture takes place.

370. Hitherto, we have supposed that there was no external force acting on the revolving particle. Let us, however, suppose that there is a resultant force P, whose components are P cos α, and P sin α, respectively parallel and perpendicular to the tangent to

the curve. Then P cos α will simply go to increase or diminish the velocity; but the equation for the centripetal force will now be

$$Mf \pm P \sin \alpha = \frac{MV^2}{r},$$

where Mf is the centripetal force, and the + or − sign is to be taken according as P sin α acts towards or from the centre of the curve.

371. If we suppose P sin α to act towards the centre, then it is evident that as long as $\frac{MV^2}{r}$ is greater than P sin α, f will still be positive; or, in other words, the particle will still remain on the curve, in spite of the inwards tendency of the impressed force. This explains how it is that a loose stone in a sling, or the water in a bucket, can be whirled round in a vertical circle without falling off. At the highest point of such a circle the tendency to fall is measured by the weight of the body, or by Mg; and the condition that the body shall not fall is given by

$$\frac{MV^2}{r} > Mg, \text{ or } V > \sqrt{gr},$$

from which the requisite velocity can be calculated.

§ 5. Elasticity.

372. In our discussions upon energy we supposed throughout that the centres of force concerned were separated from each other by a finite distance, and that the forces acting between them were actually or approximately constant. But it is clearly conceivable that two centres of force may come within an indefinitely small distance of each other—in other words, may meet; and all experience confirms what is expressed in the definition of matter, namely, that the forces between the centres, as actually existing, are not constant, but vary with the distance. It becomes therefore necessary to consider what will happen if two centres meet;

and to do this we must examine more closely what the laws of the forces acting between them really are.

373. Let us, as a preliminary, examine what would happen in the meeting of two equal centres, which attract each other, if the forces were really constant. Suppose them to start from rest, at a distance from each other of 2 feet. Then, by the principle of symmetry, they would meet at the midway point, each with a finite velocity due to the action of the constant attractive force over the space of 1 foot. There being nothing to stop this velocity, they would pass through each other—in this ideal state of things we need not discuss the question of penetrability—and go forward each in its old direction. But the attractive force on either centre, being now in the opposite direction to that of motion, would check this velocity, and destroy it at the end of the same space in which it was generated—*i.e.*, 1 foot. Hence, when each centre had arrived at the precise spot originally occupied by the other, it would be at rest. The circumstances would now be the same as at first; the centres would begin to approach each other again, would again pass through each other, and return to their original positions. This process would go on for ever, the two centres describing regular oscillations about the midway point.

374. Let us now make another supposition. Suppose that, when each of the centres had moved half way to the middle point, or through 6 inches, the constant attractive force was suddenly changed to an equal and opposite repulsive force. This would destroy the velocity thus acquired in exactly the same space in which it was generated. Consequently, at the instant when the two particles met each other, they would both come to rest. The repulsive force would then drive them asunder; but if, when the two were once more 1 foot apart, it was changed back into an attractive force, the velocities would be again checked, and the centres would come to rest precisely in their original positions. They would then again approach each other as before, and would thus continue to oscillate backwards and forwards, alternately approaching to and receding from the midway point.

375. It is needless to say that nothing approaching either of these processes has ever been observed; and, in fact, we know that

the forces of the universe are not constant. At the same time the two cases illustrate clearly a way in which a stable or conservative movement—or an oscillatory movement, using the word in its most general sense—may be produced by the action of attractive forces, or of attractive and repulsive forces combined; and we know that the world is, on the whole, in such a stable condition.

376. If then, the forces which hold in nature are not constant, what are the laws according to which they vary? Unfortunately, the present advance of mechanical science does not enable us to answer this question. There is, indeed, an answer which appears to be complete and accurate (within our present limits of observation), so long as the points are at a considerable distance from each other; but it is certainly not complete when they approach within a certain distance. The law here referred to is that discovered by Newton, and known as the law of gravitation. Expressed in the strictest terms, it is as follows:—*Every centre of force in the solar system attracts every other centre with an equal force, varying inversely as the square of the distance between them.*

By equal forces are meant, of course, forces which are equal at equal distances.

377. The above statement refers to individual centres of force. Let us now extend it to the case of two bodies, A and B, which are so far apart that their own dimensions may be neglected in comparison with the distance between them. Let this distance be r, and let n n' be the number of centres in the two bodies respectively. Let f be the absolute value of the force—that is to say, the accelerating force acting between two centres when at a unit of distance apart. Then the attraction which any given centre in A exercises upon any given centre in B is expressed by $\frac{f}{r^2}$. The sum of the attractions which it exercises upon all the n' centres in B will therefore be expressed by $n'\frac{f}{r^2}$. And this sum will be the same (approximately) for each one of the n centres in A. Hence the total attraction exercised by A upon B will be given by $n\,n'\frac{f}{r^2}$.

And by the definition of matter, the attraction exercised by B upon A will be equal and opposite to the above, and will be represented therefore by $-nn'\frac{f}{r^2}$. But, by Art. 60, n, n' are proportional to the masses of the bodies. Let these be m, m'. Then the total attraction may also be written $mm'\frac{c}{r^2}$, where c is the attraction between two bodies of unit mass at unit distance. Expressing this in words, and remembering the definition of a particle as a body so small that its dimensions may be neglected, we may state the law of gravitation as follows :—

Any particle in the solar system attracts any other particle with a force which varies jointly as their masses, and inversely as the square of the distance between them.

This is substantially the form in which the law is usually stated.

378. To give the proof of this law is beyond the scope of the present treatise. Assuming it to be true—as all competent judges assume—we have next to inquire whether it is the only law; in other words, whether it will, by itself, account for all the phenomena of the universe. As already stated, this must be answered in the negative. For, were two centres left to themselves under this law, they would rush together with a velocity which, at the instant of meeting, would become infinite, since, the distance being nothing, the force would then be infinite. What would happen it is needless to inquire, but at least it would not be anything like what we see around us. Nor is the case altered by the existence of other centres. A system starting from rest, under the action of gravitation alone, would coalesce in like manner. Hence there must be something beyond gravitation—something which acts as a repulsive force, and prevents the centres from thus dashing themselves against each other. Unfortunately, we cannot give the law, scope, &c., of this force as we can in the case of gravity; but we may glean a few facts respecting it. It must be practically insensible at sensible distances; otherwise, the law of gravity would not be found fully to account for the facts of astronomy and of falling bodies, which it is known to do. Hence it must diminish as the distance decreases; in other words, it must vary

inversely as some power of the distance. But this power must be higher than the square; for if it were less than the square, it would increase faster than gravity; and if it were the square itself, it would increase or diminish just as fast as gravity, and no faster, and would only have the effect of diminishing the apparent absolute value of gravity. These conditions are satisfied by assuming that the real law of force acting between two particles is represented, not by the expression $m m' \frac{c}{r^2}$, but by the fuller expression

$$m m' \left(\frac{c}{r^2} - \frac{c'}{r^{2+n}} \right],$$

where n is some positive quantity.

379. It may perhaps be thought that the second term of this expression could not possibly disappear from view so completely as it does in all questions relating to the attraction of gravity at sensible distances. To examine this point, let us suppose that the attractive and repulsive forces are equal in amount at a distance of one-millionth of an inch, and also that the repulsive force varies as the fourth power of the distance. Then we have—

$$c \times (1{,}000{,}000)^2 = c' (1{,}000{,}000)^4$$

$$\text{or } c = c' (1{,}000{,}000)^2.$$

If the distance is one-thousandth of an inch, the expression becomes—

$$m m' [c (1000)^2 - c (1000)^4$$

$$\text{or } m m' \left[c (1000)^2 - c (1000)^2 \times \frac{1}{1{,}000{,}000} \right).$$

Hence, at the distance of one-thousandth of an inch, the repulsive force will be only one-millionth part of the attractive force, and therefore quite insensible.

The assumptions here made are, of course, arbitrary; but they are sufficient to show how easily the repulsive force may really exist at all distances, yet may be imperceptible, by the most delicate measurements, unless at distances which are almost inconceivably small.

380. Assuming the law of force to be something like what has been described, let us now consider what will happen when two centres of force are left to its operation. We may consider the motion of one of them, B, relatively to the other, A, taken as fixed.

381. Suppose, first, that B is placed exactly at the point of equilibrium, that is, at the point where the attractive and repulsive forces balance each other. Then B will clearly remain at rest.

382. Suppose next that B is slightly beyond this point of equilibrium. Then, the attractive force being slightly the largest, B will move towards A, and will pass the point of equilibrium with a certain small velocity. From this moment, however, the repulsive force will be the largest; the velocity of B will consequently be checked, and at a certain small distance within the point of equilibrium it will come to rest. The repulsive force being still the largest, the same operations will then begin in the reverse order; B will be repelled from A, and pass the point of equilibrium with the same velocity as before, but in the reverse direction, and will then be checked by the attracting force, and brought to rest exactly at the point from which it originally started. The same cycle will then begin again; in other words, B will continually describe, with regard to A, a series of small oscillations about the point of equilibrium. If B is placed at first slightly within this point, instead of beyond it, the same events will follow, but in the reverse order.

383. In either of the above cases, suppose a third force to act upon B, tending to move it towards A. So soon as B is within the point of equilibrium, the repulsive force will be larger than the attractive force, and the excess will increase very rapidly as B continues to approach A. Hence this excess of the repulsive force will soon counterbalance the external force, and B will remain at rest at a new point of equilibrium, defined as being the point where the three forces balance each other; or, rather, will continue making small oscillations about that point.

If, on the contrary, the third force tends to move B away from A, then the attractive force will be in excess, will counterbalance the third force, and will form a new point of equilibrium farther away from A than the original one.

In the first case, the force is *compressive*, and the net result is that, so long as the force acts, the distance between A and B will be permanently shortened. In the second case, the force is *tensile*, and the net result is that, so long as the force acts, the distance between A and B will be permanently lengthened.

384. Again, let us suppose that B starts from a point at a considerable distance beyond the point of equilibrium. Then, by the time it reaches that point, the attractive force, which throughout this distance is largely in excess, will have imparted to B a very considerable velocity. Or, which comes to the same thing, we may suppose that B starts from the point with a considerable impressed velocity towards A. As soon as B has passed this point, its velocity will be checked by the excess of the repulsive force; and it will be destroyed, and B brought to rest, somewhere in the very small space between the point of equilibrium and A. But B will never come in actual contact with A. For if A and B were in actual contact, the distance between them would be indefinitely small, and therefore the repulsive force would be indefinitely great. But the accelerating force which would destroy any given finite velocity v in any given finite distance s is simply given by the expression $\frac{v^2}{2s}$: and, however large v may be, or however small s may be, this will always have a finite value. Hence the effect will be that B will be stopped in an exceedingly short space and time—much too short for our measurement—and will then have a very large excess of repulsive force acting upon it. Hence it will begin to return with very great rapidity, will pass the point of equilibrium with the same high velocity, but in the reverse direction, and will then be checked by the excess of attractive force, finally coming to rest at the point from which it started. If no other cause intervenes, the same cycle will then begin again.

385. Lastly, let us suppose that in the last case B becomes fixed in space at the moment when it is stopped by A, while A becomes free to move; or, which comes to the same thing, let us consider the motion relatively to B, instead of relatively to A. Then, since *A, by the* third law of motion, has exactly the same repulsive

force acting upon it as B has, it will fly off with the same rapidity as was ascribed to B in the last section, will travel to exactly the same distance, and there will come to rest and begin to return, unless some other cause supervene.

386. These deductions from the assumed form of force have been seen to follow naturally and clearly upon each other. It now remains to show that they accord with the facts of the universe, as relates to the behaviour of the particles of solid bodies in close contact with each other. Of course, the matter is greatly complicated by the fact that we can never observe the motions of single centres of force, or even single particles. What we observe are bodies, greater or less in size, but of which the adjacent particles act in various ways upon each other, and are also acted upon in various ways by external forces, such as gravity. Nevertheless, effects similar to those here described are in many cases plainly discernible.

387. Thus, in accordance with Art. 382, the particles of any solid body do take up apparent positions of equilibrium with each other—which, however, are known not to be really positions of rest, but centres of small oscillations which the molecules are continually describing.

388. If an external force be brought to bear upon such a body, then, in accordance with Art. 383, the body becomes extended if the force be tensile, or shortened if the force be compressive; and, having thus taken up a new position of equilibrium, it retains it until the force is withdrawn. Should the force be beyond the resisting powers of the body, the extension or compression goes on until fracture of some sort takes place.

389. Again, if a body be projected against another with considerable velocity, which is equivalent to the supposition of Art. 384, then, after apparently striking it, it flies back in the direction from whence it came, the reversal of the motion being usually effected far too rapidly for any ordinary means of observation to follow it. It is this property of rebounding which forms what is called the *elasticity* of bodies. Newton, who investigated it, found that the effects might be represented by supposing that, at the moment of

impact, the momentum of the striking body was stopped by a very large force, brought into existence by the action, and opposing the original force of impact; and that the bodies having thus been brought to rest, the force continued to act in the same direction, and to drive the striking body back again towards the point whence it started. This agrees fully with Art. 384. If the body actually reaches that point, it is called "perfectly elastic." As a matter of fact, no substance in practice is found to be perfectly elastic; some, however, as glass and ivory, approach the limit pretty closely, while others can scarcely be said to have any visible elasticity. For some time this was supposed to show that in practice the force of restitution, causing the rebound, was somehow less than the original force of impact, in a varying ratio, generally expressed by the letter e. But it is now universally admitted that this difficulty simply arises from the fact that we can only observe the action of finite bodies as a whole, and not that of their minute parts. For instance, when a billiard ball strikes the cushion, it is but a very small area of each which is actually opposed to the other; all the rest of the billiard ball is caused to stop and to rebound by the lateral cohesive actions of the parts nearer the centre. These actions set up movements between the particles, in which more or less of the energy due to the impact becomes expended, and is not therefore available for the repulsion of the body as a whole. It is not now doubted that the ultimate atoms of any body are perfectly elastic.*

390. Lastly, if the body struck be free to move, instead of being fixed, the result stated in Art. 385 is actually seen to follow—that is, the struck body flies away with a velocity which depends on the momentum of the striking body; or, in other words, upon the force of the blow. Familiar instances are the striking of one billiard ball by another, the propulsion of a football, &c. In such

* In other cases the energy is largely expended in "deforming" or altering permanently the shape of the body, as a lump of clay is flattened by falling to the ground. The power of bodies to resist this, or to recover their form, is sometimes called their elasticity; but in mechanics the term *merely indicates* the existence of a repulsive force causing a rebound.

cases the striking body may either be brought to rest, or may follow in the same direction as the struck body, but with diminished speed, or may rebound again in the direction whence it came. These variations depend upon variations in the masses and velocities of the two bodies, as will be seen hereafter.

391. It appears therefore that the fundamental facts of elasticity are all accounted for by the hypothesis that the law of the force subsisting between any two centres is substantially of the character represented by $\left(\frac{c}{r^2} - \frac{c'}{r^{2+n}}\right)$. Of course, we do not affirm that this is its exact representation. It may be much more complicated—*e.g.*, there may be other factors which may bring about the differences existing between the chemical elements, considered as kinds of matter.

392. Again, the centres of force are in nature grouped together permanently into separate atoms or molecules (Art. 52). It is further probable that the centres comprised in each molecule are in very rapid motion relatively to each other, and that each molecule is also in rapid motion with reference to other molecules. All these facts tend to complicate immensely the estimation of the forces acting in any particular case; but the fact remains that the general law of force must be, speaking roughly, of the character indicated above.

393. From this discussion we deduce the following definitions and laws:—

Definition.—*A body is elastic when, after striking another body considered as fixed, it rebounds, or receives a motion in the opposite direction to the original motion.*

Definition.—*A body is perfectly elastic when, the impressed forces remaining the same, the motion of rebound is in all respects the reverse of the motion of impact.*

Definition.—*A body is imperfectly elastic when the motion of rebound is less than that of impact.*

First Law of Elasticity.—*The ultimate atoms of every body are perfectly elastic.*

Second Law of Elasticity.—*Every body, taken as a whole, is imperfectly elastic.*

Third Law of Elasticity.—*In any body, taken as a whole, and therefore imperfectly elastic, the ratio which the force of restitution bears to the force of compression depends only on the nature of the body.*

§ 6. Impact.

394. It has been said that when one body strikes and rebounds from another, the action is usually so rapid that it cannot be directly measured. It should be observed, however, that this is not always the case. The difference is well illustrated by the mineral waggons on a railway, some of which have dead buffers—that is, buffers composed of wood,—while others have spring buffers—that is, composed of an iron head and bar, packed with discs of india-rubber. If two of the former strike each other, they will be seen to spring back instantaneously, so far as the eye can judge; but if two of the latter strike each other (at a moderate velocity), there will be an appreciable time during which the buffers are yielding before they come to rest, and, again, an appreciable time before they completely separate.

395. In such cases as the last, if the law of resistance of the spring is known, the time and distance in which the body will be brought to rest can be determined by the usual formulæ for accelerating force, in the same way as we determined the stoppage, under the action of gravity, of a body projected vertically upwards.

Example.—Two railway waggons, each weighing 12 tons, meet each other at a speed of 8 feet per second, and after the buffers touch, each waggon moves 6 inches before coming to rest: To find the mean resistance of the buffers—that is, the force of resistance supposing it to be constant throughout the collision.

The energy of either waggon at the moment when the buffers touch is given by the expression $\frac{w}{g}\frac{v^2}{2}$, or (taking $g = 32$)

$$\frac{12 \times 2240}{32} \times \frac{8 \times 8}{2} = 12 \times 2240,$$

and this is destroyed by the exertion of an energy Rs, where R is *the resistance*, s the space through which it acts, or ½ foot. Hence

$$R \times \tfrac{1}{2} = 12 \times 2240;$$
$$R = 24 \times 2240 \text{ lbs., or } 24 \text{ tons.}$$

396. In cases where the action is very much more rapid, the same treatment may still be used, provided we may make certain assumptions as to the conditions. For example, suppose that the waggons in the last case had dead buffers, and that the wood yielded by the sixteenth of an inch before they came to rest. Then we have only to substitute in the equation $\frac{1}{12 \times 16}$ for $\frac{1}{2}$, and the value of R is given by

$$R = 12 \times 16 \times 12 \times 2240 \text{ lbs.} = 2304 \text{ tons.}$$

The accelerating force of this pressure upon the waggon is $\frac{R}{w} \times g$, or

$$12 \times 16 \times 32 = 6144 \text{ feet per second.}$$

The time occupied in stopping the motion is the time in which this accelerating force will generate a velocity of 8 feet per second. Hence, since $v = ft$, we have

$$\text{time} = \frac{8}{6144} = \frac{1}{768} \text{ of a second.}$$

This example will suffice to show how rapid is the action and how great the forces called into play, even in the case of impact at very moderate velocities.

397. In practical cases, where the times and spaces cannot be exactly observed, the forces of restitution must be treated as impulsive forces—that is (Art. 356), they must be estimated according to the total momenta which they generate. It is to questions of this character that the title of impact is usually confined.

Let us first consider the impact of totally inelastic bodies.

Problem.—*Two inelastic balls, moving in the same direction, but with different velocities, impinge upon each other; to determine the motion after impact.*

398. Let M M′ be the masses of the two balls, V V′ their velocities before impact.

When the bodies impinge, there will be an impulsive pressure between them, which we will call R, and which will be measured

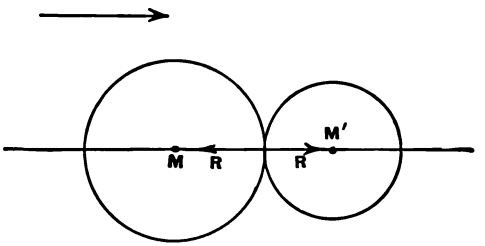

Fig. 65.

by the momentum it generates. This pressure will be equal in magnitude and opposite in its direction upon the two balls—*i.e.*, accelerating one and retarding the other. For while the pressure is acting, the bodies are in contact, and therefore moving with the same velocity: hence the conditions are the same as if they were at rest, and R will be the sum of the forces acting between the bodies, which by the third law of motion will be the same for each body.

The momentum of the ball M before impact is MV; therefore its momentum after impact is $MV - R$, and therefore its velocity is $V - \frac{R}{M}$.

Similarly, the velocity of the ball M′ after impact is

$$V' + \frac{R}{M'}.$$

But since, by the hypothesis of inelasticity, there is no force after impact to separate the balls, they will proceed with a common velocity;

$$\therefore V - \frac{R}{M} = V' + \frac{R}{M'};$$

$$\therefore R = \frac{MM'}{M + M'}(V - V');$$

and if v be the common velocity,

$$v = V - \frac{M'}{M+M'}(V - V') = \frac{M\,V + M'\,V'}{M+M'}.$$

399. **Cor. 1.**—If the balls are moving in opposite directions, we have only to write $-V'$ instead of V'. Then the equations become

$$R = \frac{M\,M'}{M+M'}(V + V'),$$

$$v = \frac{M\,V - M'\,V'}{M+M'}.$$

400. **Cor. 2.**—In the case of Cor. 1, if $V\,M = M'\,V'$, we have $v = 0$. In other words, if two inelastic balls, moving in opposite directions, and with equal momenta, impinge on each other, they will be reduced to rest.

401. **Cor. 3.**—From the two equations for v we have, according as V' is positive or negative,

$$M\,v + M'v = M'\,V + M'\,V',$$
$$\text{or } M\,v + M'\,v = M'\,V - M'\,V'.$$

Both of these are expressed by saying that *the algebraical sum of the momenta of the balls is the same after impact as before it.*

402. We can now proceed to the problem of elastic bodies. We may consider the impact as consisting of two parts—viz., during the compression of the bodies, and during the restitution of their forms. As long as compression continues, the problem is precisely the same as if the bodies were inelastic; and if we call R the impulsive pressure between them during compression, the value of R will be that already found on the supposition of the bodies being inelastic. For, though the bodies do not, for any sensible time after impact, move with the same velocity, yet during that very short time in which the compression takes place they do so; hence the *force of compression* is already determined. When the restitution of form takes place, a new force R' is brought into action, which we have distinguished as the *force of restitution*. To determine R' we must have recourse to experiment, and it is

found (Art. 389 above) that the ratio of R′ to R is independent of the velocity of the bodies, and dependent only on the nature of the substances of which they are composed. So that, if we make $R' = e\ R$, we may consider e to be a known quantity; since in any given example, if the substance of the bodies is given, the value of e may be found from experiment, or by reference to tables of elasticity. The quantity e is called the *modulus of elasticity*; for finite bodies it is always some quantity less than 1.

403. **Problem.**—*Two elastic balls moving in the same direction, but with different velocities, impinge upon one another; to find the velocities after impact.*

Let MM′ be the masses of the bodies,

VV′ their velocities before impact,

vv′ „ after „

RR′ the forces of compression and restitution respectively, so that the whole impulsive force between the balls $= R + R' = R(1 + e)$, where e is the modulus of elasticity.

We may find R on the supposition of the bodies being inelastic; hence by our previous investigation (Art. 398),

$$R = \frac{M M'}{M + M'}(V - V');$$

$$\therefore v = V - \frac{R + R'}{M} = V - (1 + e)\frac{R}{M} = V - (1 + e)\frac{M'}{M + M'}(V - V'),$$

$$v' = V' + \frac{R + R'}{M'} = V' + (1 + e)\frac{R}{M'} = V' + (1 + e)\frac{M}{M + M'}(V - V').$$

404. **Cor. 1.**—If the balls are moving in opposite directions we must change the sign of V′; then the equations are,—

$$v = V - (1 + e)\frac{M'}{M + M'}(V + V'),$$

$$v' = V' + (1 + e)\frac{M}{M + M'}(V + V).$$

405. **Cor. 2.**—We have, by subtraction,

$$v - v' = V - V' - (1 + e)(V - V'),$$

$$= -e(V - V').$$

If we suppose V to be greater than V′, and that after the impact the ball M′ is driven on by M in the direction in which it was moving before impact, v' will be greater than v, and we may write the preceding equation thus,

$$\frac{v' - v}{V - V'} = e.$$

Now $V - V'$ is the *relative* velocity of the balls before impact, that is, the rate at which they approach each other, and $v' - v$ is the relative velocity after impact, or the rate at which they separate; hence the preceding formula may be expressed by saying, that the ratio of the relative velocities before and after impact is a quantity depending only on the nature of the substances of which the balls are composed.

406. **Cor. 3.**—If we multiply v by M, and v' by M′, and add, we have, according as V′ is positive or negative,

$$M v + M' v' = M V + M' V',$$
$$\text{or } M v + M' v' = M V - M' V'.$$

Hence, as before, the algebraical sum of the momenta of the balls is the same after impact as before it.

This result, which is thus shown to be general, might be deduced by general reasoning. For it appears that the total action on either ball, whether inelastic or elastic, varies as R, where R is an impulsive pressure, acting equally in opposite directions on the two balls, and measured by the momentum which it generates. Hence the effect of the impact is to generate in the two balls, considered as one system, an equal amount of momentum in opposite directions; and it is clear that this cannot have any effect on the momentum of the system taken as a whole.

407. **Problem.**—*An elastic ball impinges directly upon a fixed plane; to find the velocity after impact.*

Let V be the ball's velocity before impact,
v ,, after ,,
R R′ the forces of compression and restitution,
e the modulus of elasticity.

Then, to find R, we suppose the body inelastic; but in this case there would be no velocity after impact, since the plane is fixed;

$$\therefore V - \frac{R}{M} = 0, \text{ or } R = MV;$$
$$\therefore R + R' = (1+e)\,MV,$$
$$\text{but } Mv = MV - (R + R')$$
$$\therefore v = V - (1+e)\,V = -eV.$$

Hence the ball's velocity will be diminished in the ratio of $1 : e$. The negative sign indicates that the motion after impact must be in the opposite direction to that before impact, which must manifestly be the case.

408. By the term *oblique* impact we designate those cases of impact in which the direction of the velocity does not coincide with the direction of the mutual impulsive pressure.

Problem.—*A body impinges upon a fixed plane, in the direction of a line making a given angle with the normal to the plane: to determine the motion after impact.*

Let V be the velocity before impact, α the angle which its direction makes with the normal to the plane: v, θ similar quantities after impact. The rest of the notation as before.

We may suppose the velocity V to be resolved into two velocities, one parallel to the plane ($V \sin \alpha$), the other perpendicular to it ($V \cos \alpha$); the former will not be altered by the impact, the latter may be treated as in the case of direct impact, and will therefore be diminished in the ratio of $1 : e$. The resolved parts of the velocity after impact, parallel and perpendicular to the plane, are $v \sin \theta$, and $v \cos \theta$ respectively; hence we shall have,

$$v \sin \theta = V \sin \alpha,$$
$$v \cos \theta = -eV \cos \alpha;$$
$$\therefore \cot \theta = -e \cot \alpha, \text{ and } v^2 = V^2 (\sin^2 \alpha + e^2 \cos^2 \alpha),$$

which equations determine θ and v.

It may be observed that this investigation is applicable to the *case of impact* on any surface, by substituting for the plane on

which the impact has been supposed to take place the plane which touches the surface at the point of impact.

409. **Cor.**—If the elasticity be perfect, or $e = 1$, we shall have

$$\cot \theta = -\cot a,$$

$$\text{or } \theta = -a,$$

$$\text{and } v^2 = V^2, \text{ or } v = V.$$

The interpretation of these results is, that the ball will rebound from the plane with a velocity equal to that of incidence, and in a direction making an angle with the normal equal to the angle of incidence, but on the opposite side of the normal. This is the ordinary rule in the case of a billiard ball striking the cushion.

410. The more general case of the oblique impact of two balls may be solved in like manner by resolving the velocity of each ball into two, namely, one in the direction of the mutual impulsive pressure, and the other in the direction at right angles to it; then the latter portions of the velocities will not be affected by the impact, and the former will be modified exactly in the same way as if the impact had been direct.

411. We can now prove mathematically what we have already mentioned—viz., that the conservation of energy does not hold in the case of imperfectly elastic bodies, inasmuch as the sum of their kinetic energies (or the total *vis viva*) is always diminished by the impact. We have already seen in fact (Art. 400), that if the bodies be totally inelastic, and if they meet each other with equal momenta, the velocities, and, therefore, the actual energies, will be totally *destroyed* by the impact; and if the elasticity be imperfect, a similar effect will, of course, take place, though in a less degree. We shall, however, give a direct proof.

412. **Theorem.**—*In the direct impact of perfectly elastic bodies, the sum of the masses of the bodies multiplied each by the square of its velocity is the same before and after impact.*

Let M M′ be the masses of the bodies, V V′ their respective velocities before, and v v' after, impact.

Then, we have seen (Art. 403) that (taking $e = 1$)

$$v = V - \frac{2M'}{M+M'}(V - V'),$$

$$v' = V' + \frac{2M'}{M+M'}(V - V'),$$

$$\therefore v - v' = V - V' - 2(V - V') = -(V - V'),$$

$$\text{or } v + V = v' + V' \quad . \quad . \quad . \quad . \quad (1.)$$

Again,

$$M v + M'v' = M V + M'V',$$

$$\text{or } M(v - V) = -M'(v' - V') \quad . \quad . \quad . \quad (2.)$$

Multiplying together (1) and (2), we have

$$M(v^2 - V^2) = -M'(v'^2 - V'^2),$$

$$\text{or, } M v^2 + M'v'^2 = M V^2 + M'V'^2.$$

But the mass of a body multiplied by half the square of its velocity is its *Vis Viva;* hence it appears that when the elasticity is perfect, the total *Vis Viva* of two impinging bodies is not altered by impact.

413. Theorem.—*In the collision of imperfectly elastic bodies,* Vis Viva *is lost by the impact.*

In this case we have

$$v = V - (1+e)\frac{M'}{M+M'}(V - V'),$$

$$v' = V' + (1+e)\frac{M}{M+M'}(V - V');$$

$$\therefore Mv + M'v' = M V + M'V';$$

also $v - v' = V - V' - (1+e)(V - V') = -e(V - V')$;

$$\text{thus } (Mv + M'v')^2 = (M V + M'V')^2,$$

and $M M'(v - v')^2 = M M'e^2 (V - V')^2$

$$= M M'(V - V')^2 - (1 - e^2) M M'(V - V')^2;$$

$$\therefore \text{ by addition, } (M+M')\ (Mv^2+M'v'^2)=(M+M')\ (MV^2+M'V'^2) - (1-e^2)\,MM'\,(V-V')^2,$$

$$\text{or } Mv^2+M'v'^2=MV^2+M'V'^2-(1-e^2)\,\frac{MM'}{M+M'}(V-V')^2;$$

which proves the proposition, since e is less than 1.

As already mentioned, this does not affect the truth of the Conservation of Energy, inasmuch as the ultimate atoms of all bodies are perfectly elastic.

§ 7. Energy and Work.

414. In the remainder of this treatise we shall develop somewhat further the principles of Work and Energy, as laid down in Part I., Sections 10 and 11. We begin by returning to the Conservation of Energy.

415. On account of the importance of this principle, we preferred to give a formal proof of it (Art. 162), following on the general principles of Geometry and Dynamics.[1] But we must repeat that it is an immediate deduction from the principle of conservation (namely, that effects live), as soon as we have proved (as in Art. 136) that the true measure of the effect of a force acting upon any particle for any time, is the change which it has wrought in the energy of the particle, taking the word to include both the potential and kinetic energy. For this effect cannot disappear, except by producing an equivalent effect in some other body; or, which is the same thing, the energy of the particle cannot be diminished, except the energy of some other particle or particles be increased by a like amount. And this is what is meant by the conservation of energy.

416. Problem.—*To find the symbolical expression for the conservation of energy, in the case of a single particle in motion under the action of given forces in one plane.*

[1] The proof usually given (see, *e.g.*, Routh's *Rigid Dynamics*) is more elegant and complete, but involves a higher application of the Differential Calculus than is admissible in this treatise.

In this case the other particles of the system are supposed at rest, and therefore the kinetic energy of the system is given by that of the moving particle m. Let this start from the origin, and move through a very small distance, r, in a very small time, dt. Let the projections of r upon the axes be dx, dy; and let X, Y be the resultant of all the forces acting on the particle, when resolved in those directions. Then the total energy expended on the particle is found by multiplying each of these forces by the distance through which the particle has moved in its direction, and taking the sum. This sum is clearly

$$X\,dx + Y\,dy.$$

Since this represents the energy expended by the forces, it also represents the change in the potential energy of the system (apart from the moving particle) which is due to the motion.

Again, let v be the velocity of the particle at the beginning of the interval, and $v+dv$ at the end. Then the change in the kinetic energy of the particle is given by the expression

$$\frac{m}{2}\Big[(v+dv)^2 - v^2\Big].$$

And these two changes, by the principle, are equal to each other. Hence we have

$$\frac{m}{2}\Big[(v+dv)^2 - v^2\Big] = X\,dx + Y\,dy.$$

[417. If the motion is in three dimensions, we may similarly deduce the equation

$$\frac{m}{2}\Big[(v+dv)^2 - v^2\Big] = X\,dx + Y\,dy + Z\,dz].$$

418. There are, of course, no instances in nature where all the particles of a system but one are really at rest; but there are many instances where all the particles, whose influence on a certain particle needs to be taken into account, may be considered as at rest with respect to it. Such cases are those where gravity is the only force acting. Thus, suppose a body to fall vertically *from rest* under the action of gravity, and let v be its velocity after

it has fallen through a space s. Then the left hand side of the equation becomes $\frac{m}{2}v^2$, and the right hand side becomes mgs. Hence the equation becomes

$$m\frac{v^2}{2} = mgs,$$

which is equivalent to the well-known formula, $v^2 = 2gs$, given in Art. 90. In this case the above equation holds exactly; that is, the potential energy of the body is diminished (since it has approached nearer to the centre of the earth), while its own kinetic energy has increased by a corresponding amount.

Again, let us take the case of a particle projected upwards, with initial velocity V. Then the equation becomes

$$m\frac{V^2 - v^2}{2} = -mgs.$$

Here the potential energy expended is represented by the *negative* quantity $-2gs$; in other words, the potential energy has been *increased* by the quantity mgs. At the same time the kinetic energy has been *diminished* by the corresponding amount.

419. It is only in very simple cases that the whole change of energy, from potential to kinetic, thus takes place, as it were, within the moving body itself. Generally the energy is expended, wholly or in part, on other bodies, which therefore have to be taken into account. In such cases we must subtract from the total energy expended by the moving forces the energy which has been expended upon these other bodies; and the remainder will represent the energy which has been expended on the moving body, and which must therefore be equated to the change in the kinetic energy. The energy expended on the other bodies will have gone to increase either their actual or potential energy: this will have to be a matter of separate inquiry.

420. Thus, suppose the case of a train starting from rest on a level road. Let P be the forward force exercised upon it by the engine, and let R be the resistance to motion, which we may, for the present, consider constant. Let m be the mass of the train, so that

velocity after passing over a distance x. Then the total energy exerted by the engine is Px. Also, since the resistance, R, has been moved through the same space x, the energy expended on the resistance is Rx. Hence the equation is

$$\frac{m}{2}v^2 = Px - Rx.$$

421. Let us consider what has become of the energy, Rx. Of this the greater part has been expended in overcoming the friction between the axles and their bearings, and between the tires of the wheels and the rails. It is not at first sight easy to see where this energy has gone, and formerly it was supposed to be altogether lost. It is, however, a well-known fact that the bearings of railway axles, if not attended to, become so hot as to set fire to the grease used for lubricating them; and the tires may also become heated. From such facts it was surmised long ago that friction was mainly absorbed in producing heat; and it is now proved, beyond all reasonable doubt, that the condition of bodies which produces the sensation we call heat, is a condition of rapid vibratory motion in the molecules which compose them. The heat of a body, in fact, is measured (within certain limits) by the kinetic energy, or *vis viva*, of its molecules. Hence we see that the greater part of the energy, Rx, has been expended in increasing the kinetic energy of the particles in the bearings, axles, &c. Another part has been expended in overcoming the resistance of the air—that is, in pushing aside its particles and giving them motion, which motion may be felt by a bystander as a decided breeze. This part, therefore, has also gone in increasing kinetic energy—namely, that of the air. Lastly, if the train, instead of being on a level, is on an ascending gradient, the resistance offered by the weight of the train to the ascent of this gradient must be included in the value of R; and the energy thus expended goes to increase the potential energy of the train, since, if stopped and left to itself, it would roll back again down the gradient, thus increasing its kinetic energy.

422. We have hitherto supposed the resistance R to be constant. *If so,* P being always greater than R, the velocity of the train would *increase* indefinitely. As a matter of fact, we know that this is not

so. However powerful the engine, the train at last reaches a certain speed, which it cannot surpass. It follows that the resistances are not constant, as is found to be true on experiment. The frictional resistances indeed decrease slowly as the speed increases, at any rate after a certain limit; but the resistance of the air increases very rapidly (about as the square of the velocity), and it is this which brings the increase of speed to an end. When this takes place, the train continues to move with a constant velocity, under the action of a forward or accelerating force, P, and a retarding force, also equal to P. It is, therefore, in a state of equilibrium, and might be treated as if it were at rest. At the same time, it would not be true, of course, to say that no work was being done in this case. The force, P, of the engine exerts energy just as before; but the whole of this energy is finally expended, not on the train, but on the other bodies, chiefly the particles of the air, to which it communicates motion, thereby increasing their kinetic energy. The train only serves, as it were, as an intermediary, through which this energy is conveyed to the air; just as the "draw-bar," which connects the engine to the train, serves as an intermediary to convey the energy to the train.

423. Hitherto we have considered the question of the transformation of potential into kinetic energy. Let us now suppose that the train is to be stopped. Then its kinetic energy, represented by $\frac{m}{2} v^2$, has to be disposed of, and it must be expended in generating kinetic or potential energy in some other bodies. If the length of the stop is of no consequence, we may simply turn off the steam, and the train will expend its energy gradually in overcoming the resistances, R—in other words, in increasing the kinetic energy of the bearings and axles, the particles of the air, &c. If, however, the train has to be stopped quickly, it is usual to apply *brakes*. These consist of shoes or blocks, generally of wood, which are pressed hard against the tires of the wheels, and exercise a retarding effect by their friction. The theory of such brakes affords a very instructive example of the absorption of kinetic energy, and also of the advantage gained by using the artifice of Art. 338, i.e.,

that of supposing a certain velocity to be impressed on all the parts of a system, so as to bring one particular part of it to rest.*

424. It should be stated here that the coefficient for dynamical friction is always very much less than the corresponding coefficient for statical friction. According to the experiments of Morin, the coefficient for dynamical friction is independent of the velocity; but it appears that this is only an approximation to the truth with moderate pressures and speeds. At high speeds, and with such pressures as those of brakes, the friction decreases as the speed increases; but the exact laws of its action are unknown.

425. Let us confine ourselves, for the sake of simplicity, to a single wheel of a railway train moving at a uniform speed. This wheel, while rolling upon the rail, revolves round its own centre with an angular velocity such that the linear velocity of its circumference is equal to the speed of the train. Any point of the wheel will thus have a motion compounded of the general horizontal motion of the train and of this rotary motion round the centre. To get rid of this compound motion, it will be well to suppose a velocity equal and opposite to that of the train to be impressed upon every point of the wheel and of the rail. On this supposition the centre of the wheel will be stationary in space; every other point of the wheel will revolve round the centre with the same velocity as before; and the rail will move with a motion the same as that of the train, but in the opposite direction. The effect is, in fact, the same as if we supposed the engine to be employed not to pull the wheel over a stationary rail, but to pull the rail from under a stationary wheel. Let us now consider what will happen. If we neglect all friction of journals, &c., and suppose that the brake is not applied, the power required to keep up the motion will be *nil*. Let us now suppose the brake applied with a pressure P. This will produce by friction a force (say fP, where f is the coefficient of friction) tangential to the wheel, and tending to stop its rotation. This force, transmitted through the frictional resistance or adhesion between the wheel and the rail, will act upon the rail, and tend to stop its motion. If this motion is to be kept

*This theory was first given before the Institution of Mechanical Engineers, Oct. 1878.

up as before, a force equal and opposite to fP must be applied by the engine to the rail. Hence, the additional tractive force required when the brake is applied (or in other words, the retarding effect of the brake) is equal to the tangential frictional stress of the brake upon the circumference of the wheel.

426. This conclusion is not strictly true except where the speed is kept uniform. When the train is stopping under the friction of the brake, a part of this friction is employed in checking the rotation of the wheel to correspond with the checking of the train. This part of the frictional resistance is thus wasted, as far as stopping the train is concerned: but it is always in practice a small fraction only of the total resistance; and it is constant at all speeds, since the rate at which the velocity of a body is destroyed depends only on the mass of the body and the amount of force applied to it, and not on the initial velocity of the body.

427. If the adhesion of the rail and wheel were unlimited, this would be a complete account of the whole matter. But this adhesion has a limit, say F W. where F is the coefficient of static friction between the wheel and the rail; F will always be much greater than f, which is the coefficient of dynamic friction between the wheel and the brake-block. Let the pressure, however, be so much increased, that the brake friction fP is greater than the rail friction F W. Then, since the force transmitted from the rail to the wheel cannot be greater than F W, it follows that, whatever be the pull of the engine, the difference between these two frictional resistances, which act in opposite directions, will remain as an unbalanced force, tending to stop the rotation of the wheel. Consequently, this rotation will be checked; and if the wheel had no inertia, it would be stopped instantaneously. As the wheel has inertia, it will be stopped, not instantaneously, but after an interval of time, which will be greater as the mass and velocity of the wheel are greater, and less as the difference between fP and F W, the two frictional resistances, is greater. In all cases, however, this interval will be very short; because, as soon as the rotation is materially checked, so that the wheel is slipping over the rail with an appreciable velocity, the coefficient of friction between wheel and rail will change from its original value for static

friction to the much smaller value for dynamic friction. Hence, whatever may have been the original difference between the two frictional resistances, it will now be largely increased; and as it is this difference which tends to stop the rotation, this stoppage will be completed in a very short time. Hence, the wheel will always continue to rotate at the train speed, until the frictional resistance between wheel and brake-block becomes greater than that between wheel and rail; but as soon as this takes place the rotation will be checked, and will be stopped completely in an interval of time which in practice will always be very small.

428. If the coefficient f of brake-block friction were the same at all speeds, the pressure which would produce skidding (or stoppage of rotation) would also be the same at all speeds; although the time it would take to skid the wheel completely would be greater at a high speed than at a low one. As the coefficient of friction is less as the speed is greater, this is not the case; and it requires a greater pressure to skid the wheel at a high speed than at a low one. But in all cases the amount of tangential brake-friction which will skid the wheel is independent of the speed, that is, unless the rail-friction varies with the speed, which there is no reason to suppose is the case.

429. Let us next consider what will happen when the wheel is skidded. Just at the moment when it is coming to rest, and the motion of the wheel under the brake-block is therefore very small, the coefficient of friction between these two will change from its value for dynamic friction to its much higher value for static friction. Consequently the frictional resistance of the brake-block will show a large increase; but this will not be transmitted to the rail, because the adhesion between the wheel and the rail is already transmitting all it can, and hence this increase in the brake-block friction will not be accompanied by any increase in the tractive force.

430. Although in stopping the rotation of the wheel, a certain amount of *vis viva* is destroyed, yet this has no retarding effect on the train; since the motion lost is only the rotatory motion of the wheel round its stationary axis, and has therefore no component *in any one* direction in space.

431. As soon as the wheel is completely skidded, the brake-block friction becomes reduced to a mere mechanical means for holding the wheel fixed, and has no longer any direct effect in stopping the train. The whole of the stopping is thenceforth done by the rail-friction. But this has now only its low value for dynamic friction, and not its high value (commonly called "adhesion") for static friction, which gave the measure of the retarding force when the wheel was in motion. Hence the retarding force available for stopping a train is greater when the wheels revolve than when they are skidded, in about the proportion of the value of the static to that of the dynamic friction between wheel and rail.

432. Let us now consider what will happen if the pressure is taken off the brake-block, and the wheel released. It is clear that as soon as the brake-block friction falls below the rail friction, the difference between the two becomes an unbalanced force tending to turn the wheel, and it will begin to rotate. The *vis viva*, or kinetic energy, thus imparted to the wheel must of course be given to it by the rail, and will produce an increased pull on the rail, in other words, an increase in the tractive force. Moreover, just when the wheel is coming to its full speed, the rail and the tyre will be coming to rest relatively to each other; consequently the friction will change from its dynamic to its static value, the pull it is capable of exerting on the wheel will be greatly increased, and the kinetic energy still wanting will be imparted to the wheel very rapidly, with of course a corresponding rapid rise in the tractive force. This will be made clear by supposing that the coefficient of friction suddenly became infinite; the wheel would then be *instantaneously* brought up to its full speed, which could not take place without a violent impulsive reaction upon the rail. Hence, when the brake is slackened on a skidded wheel, the effect will be a rise in the tractive force, gradual at first, and then very rapid, as the wheel assumes its full speed of rotation. When this is completed, the tractive force will at once fall again to the value due to the remaining friction of the brake-block and wheel, or, if the brake is taken off altogether, to zero.

§ 8. Accumulated Energy.

433. As we have seen, a body in rapid motion has a large amount of kinetic energy, and this it can be made to expend in doing work upon other bodies with which it is brought into connection. Formerly, a body in such a condition was said to have *accumulated work*, or to have a certain amount of *work stored up* in it. It is clear that this is not an accurate expression. Work is measured by foot-pounds, and only exists as it is being done. What is really stored up in the body, is the power of doing work —*i.e.*, energy; exactly as a man has stored up in his body at any moment a certain capacity for doing physical work, but cannot be said to have inside him any particular number of foot-pounds. We may, therefore, speak rightly of accumulated energy, but not of accumulated work.

434. The process of accumulating energy is very useful in cases where an intermittent source of power is used for doing regular work, or again, where the work to be done is intermittent and the power regular. Of the latter, the best example is the "accumulator," used in hydraulic machinery, for cases such as the working of docks, where the opening and shutting of lock-gates and bridges, and similar operations, have to be performed at irregular intervals of time. A steam-pump is used to force water, at a high pressure, into a large upright cylinder, having a heavily-weighted "ram" or column of cast-iron fitting into it. The water, in entering, forces the ram farther and farther out of the cylinder; so that when the ram is at the top of its stroke, we have a cylinderful of water at a high pressure, determined by the load upon the ram. When work is to be done, some of the pressure-water is drawn off from the cylinder, and passed through a water-engine, which converts its energy into mechanical work. The ram, of course, descends into the cylinder as the water escapes, and keeps up the full pressure as long as any water remains. In this case it is evident that the energy accumulated is potential energy, and is measured by the load on the ram multiplied by the height to which it is lifted.

435. The best example of the use of accumulated energy with an *intermittent* motive power is that of the ordinary fly-wheel in steam

engines. The action of a steam engine is to push a piston backwards and forwards inside a cylinder; and at the moment when the piston stops, at either end of the cylinder, it can of course be doing no work. The engine would consequently stop, and could not go on again, were there not some mode of keeping up the motion. This is usually accomplished by making the engine turn a heavy wheel, called the fly-wheel, which accumulates kinetic energy during the forward stroke, and by giving this out again, in work done on the machinery, carries the engine over these "dead-points," as they are called. This, then, is a case of the storage of kinetic, not potential energy; but it is energy, not of translatory motion, or motion in a straight line, but of rotatory motion, or motion round an axis. Therefore, to investigate the fly-wheel we must know how to measure the actual energy of a rotating body.

436. **Definition.**—*The moment of inertia of a body, taken round any given axis, is the sum of the products found by multiplying the mass of each particle of the body by the square of its distance from the axis.*

437. **Problem.**—*To find the kinetic energy of any rigid body revolving round an axis.*

Let m be the mass of any particle of the body, r its distance from the axis, and let ω be the angular velocity of the particle (Art. 335). This will be constant for every particle of the body, because it is defined to be rigid. Were the angular velocity of any particle, A, greater than that of any other, B, A would gradually gain upon and overtake B, and therefore the geometrical conditions of the body would not be constant, as by the hypothesis of rigidity they are supposed to be.

Let the particle m revolve through a very small angle, $d\theta$, in time dt. Then the linear distance through which m has moved is $r\,d\theta$; and if v be the linear velocity,

$$v\,dt = r\,d\theta.$$

But by the definition of angular velocity

$$d\theta = \omega\,dt$$

$$\therefore v\,dt = r\omega\,dt$$

$$v = r\omega.$$

Hence the actual energy of $m = \frac{m}{2} v^2 = \frac{m}{2} r^2 \omega^2$. The actual energy of the whole body will be found by taking the sum of these expressions (which are essentially positive) for every particle of the body. Hence it may be written

$$\Sigma \frac{m}{2} r^2 \omega^2 = \frac{\omega^2}{2} \Sigma m r^2$$

But $\Sigma m r^2$ is the moment of inertia of the body about the axis. *Hence the actual energy of any rotating body is given by* $\frac{I \omega^2}{2}$, where *I is the moment of inertia round the axis, and ω the angular velocity.*

438. The moment of inertia is not in most cases easy to find, without the aid of high mathematics: in the case of a fly-wheel, however, it will be sufficient in practice to treat the whole of the weight as distributed uniformly along the circumference of the circle described by the mean radius of the rim. Let r be this radius: then the moment of inertia of any particle of the wheel is $m r^2$, and if w be the total weight, the total moment of inertia is $\frac{w}{g} r^2$. Hence the total actual energy is given by

$$\frac{w}{2g} \omega^2 r^2.$$

PART V.—AXIOMS, DEFINITIONS, AND LAWS.

(Collected for reference, and to be committed to memory.)

Axioms.

1. (Art. 17.) **Principle of Counteraction.**—A force always tends to produce motion, but may be prevented from actually producing it by the counteraction of an equal and opposite force.

2. (Art. 59.) **Principle of Inertia.**—All things within our knowledge being finite, any known body under the action of any known force will only assume a finite motion in a finite time. And a body which possesses this property is said to have inertia.

3. (Art. 73.) **Principle of Conservation.**—Effects live.

4. (Art. 99.) **Principle of Symmetry.**—When a cause, or set of causes, is so related to two opposite effects, that there is no reason whatever why one of those effects should take place rather than the other, then neither of these effects will be produced by the cause or causes: and this relation is said to be a relation of symmetry.

5. (Art. 122.) **Principle of Transmission of Force.**—A force may be supposed to act at any point in its direction, provided that point be considered as rigidly attached to the point on which the force really acts.

Definitions.

1. (Art. 7.) Mechanics is the science of motion and force.

2. (Art. 10.) A thing is known to be in motion, when it is continuously changing its position in space with reference to some other thing assumed to be fixed.

3. (Art. 13.) A force is a physical cause of motion.

4. (Art. 41.) A continuous force is one which is always going on; a discontinuous force is one which acts at intervals only.

5. (Art. 49.) Matter consists of a collection of centres of force,

distributed in space, and acting upon each other according to laws which do not vary with time, but do vary with distance.

6. (Art. 60.) The number of centres of force, or the quantity of matter in a body, is called its Mass.

7. (Art. 65.) The product of the mass of a body and its velocity at any instant, is called its Momentum; and forces considered in relation to the momenta of the bodies they move are called Moving Forces.

8. (Art. 94.) When the effect of several forces, acting together on a point, is found to be the same in all respects as that of a single force supposed to act by itself on the point in their place, then this supposed single force is called the Resultant of the set of forces.

9. (Art. 95.) When the resultant of the forces, acting on a point, is zero, there is said to be Equilibrium between the forces, and the point is said to be in equilibrium.

10. (Art. 129.) When a constant force acts for any time upon a body in motion, the product of the force and of the distance through which the body has moved in the line of action of the force, is called the Energy Exerted by the force during the motion.

11. (Art. 129.) The product of half the mass of a body, multiplied by the square of its velocity at any instant, is called the kinetic energy of the body at that instant.

12. (Art. 142.) If a point be acted on by two forces, P and Q, in opposite directions, of which P is the greater, then P is called the Effort, Q the Resistance, and $P - Q$ the Unbalanced Effort.

13. (Art. 142.) Under the same circumstances, if s is the space described in any time, Ps is the Energy exerted by the effort, $-Qs$ is the Potential Work done on the point, and $(P - Q)s$ is the Kinetic Work done on the point.

14. (Art. 144.) If one centre of force, taken as fixed, acts upon a second centre, and if the second centre has a motion in the direction in which the first centre tends to move it, then the first centre is said to do Work upon the second; and the work done is measured by the product of the force, and of the distance moved through in the direction of the force.

15. (Art. 145.) Energy is the power of doing work; and the

total amount of energy possessed by any centre of force is measured by the total amount of work which it is capable of doing upon all the other centres upon which it acts.

16. (Art. 176.) A rigid body is a collection of material points or particles, the relative positions and distances of which, with regard to each other, are supposed to be absolutely fixed, so that no external force can alter them.

17. (Art. 178.) When all the points making up a rigid body move in the same direction, and with the same velocity, the motion is said to be one of Translation.

18. (Art. 179.) When of the points making up a rigid body, those which lie in one straight line are at rest, and the others move in circles round that line as an axis, then the motion is said to be one of Rotation.

19. (Art. 187.) A rigid rod, movable about a fixed point in its length, is called a Lever; the fixed point is called the fulcrum, and the parts between the fulcrum and the extremities are called the arms.

20. (Art. 188.) The Moment of a force, with respect to a given point, is the product of the force and the perpendicular from the point upon its direction.

21. (Art. 198.) A pair of equal and parallel forces, acting in opposite directions, is called a Couple: the perpendicular distance between the lines of action is called the arm; and the product of either force into the arm is the moment.

22. (Art. 199.) A straight line drawn perpendicular to the plane of a couple, and proportional in length to its moment, is called the axis of the couple.

23. (Art. 227.) Friction is the resistance opposed by a surface to the motion of a body along it.

24. (Art. 237.) The angle of repose for any two surfaces is the angle at which one of the surfaces is inclined to the horizon, when the other, being acted upon by the force of gravity only, just rests upon it without sliding.

25. (Art. 239.) A displacement of a system which is such that it does not affect the amounts or relations of the various forces acting, is called a conservative displacement.

26. (Art. 241.) If the point of application of a force be displaced through an indefinitely small space, then the displacement, as measured in the direction of the force, is called the Virtual Velocity of the force; and the product of the force and the virtual velocity is called the virtual moment.

27. (Art. 325.) A body flying through space under the action of gravity is called a Projectile, and the path which it describes is called its trajectory.

28. (Art. 334.) If the motion of a particle in a plane be considered with reference to a fixed point in that plane, the Angular Velocity of the former about the latter means the rate of increase of the angle made, by the line joining the two, with some fixed line in the plane.

29. (Art. 357.) An Impulsive force is a force so large in amount, and acting for so short a time, that we have to measure its effect simply by means of the change in velocity which it produces.

30. (Art. 363.) The Circle of Curvature at any point of a curve is that circle of which two successive elements coincide with two successive elements of the curve; and its radius is called the radius of curvature.

31. (Art. 365.) A particle moving uniformly in a circle is under the action of a constant force tending to the centre; and that force is called the Centripetal force.

32. (Art. 393.) A body is Elastic when, after striking another body considered as fixed, it rebounds, and receives a motion in the opposite direction to the original motion.

A body is perfectly elastic when, the impressed forces remaining the same, the motion of rebound is in all respects the reverse of the motion of impact.

A body is imperfectly elastic when the motion of rebound is less than that of impact.

33. (Art. 436.) The moment of inertia of a body, taken round any given axis, is the sum of the products found by multiplying the mass of each particle of the body by the square of its distance from the axis.

Laws.

1. (Art. 27.) If we can measure any interval of time, during which the velocity of a body remains constant, and also the interval of space which it travels over during that time, then the ratio of the space to the time is a proper measure of the velocity of the body.

2. (Art. 38.) Forces are measured by the velocities which they cause or generate in the same or equal objects, and in the same or equal times. This is the absolute or dynamical method; and forces when so measured are called Accelerating forces.

3. (Art. 39.) Forces may also be measured by ascertaining the number of units of a standard force (in England the number of pounds weight) which will exactly counteract them. This is the statical method; and forces, when so measured, are called Statical forces.

4. (Art. 61.) The mass of a body, that is, the number of centres of force which it contains, is measured by its weight, as compared with that of a standard body.

5. (Art. 65.) When forces act upon different objects, they are measured by the momenta which they generate in a unit of time; and such forces are called Moving forces.

6. (Art. 71.) **Newton's First Law of Motion.**—Every body continues in its condition of rest, or of uniform motion in a straight line, except in so far as it is compelled by impressed forces to change its condition.

7. (Art. 71.) **Newton's Second Law of Motion.**—Change of motion is proportional to the moving impressed force, and takes place along the straight line in which the force is impressed.

8. (Art. 71.) **Newton's Third Law of Motion.**—Reaction is always opposite and equal to action; or the mutual actions of two bodies are always equal, and in the opposite directions.

9. (Art. 76.) When any number of forces act upon a body at once, each tends to produce its whole effect in altering the magnitude and direction of the body's velocity, just as if it acted singly on the body at rest.

10. (Art. 90.) General equations for the motion of a body in one straight line, and under the action of a single accelerating force.

$$v = v_1 \pm ft \quad . \quad . \quad . \quad . \quad \text{(A.)}$$

$$s = v_1 t \pm \frac{ft^2}{2} \quad . \quad . \quad . \quad \text{(B.)}$$

$$v^2 = v_1^2 \pm 2fs \quad . \quad . \quad . \quad \text{(C.)}$$

Here f is the force, s the space described, v_1 the initial velocity, and v the velocity at the end of any time t.

11. (Art. 101.) The resultant of any number of forces acting in one straight line is found by taking the algebraical sum of the forces.

12. (Art. 115.) **Parallelogram of Forces.**—If a point is acted upon by two forces at once, whose directions make any angle with each other, the effect will be the same as if it was acted on by a single force represented in magnitude and direction by the diagonal of any parallelogram, whose sides represent in magnitude and direction the two forces acting.

13. (Art. 121.) Any force P is always equivalent to three forces acting parallel to any three rectangular axes, and having the values P cos x, P cos y, P cos z respectively, where x, y, z are the angles which P's direction makes with the axes respectively.

14. (Art. 122.) **Parallelogram of Velocities.**—The velocity of a point at any moment may be supposed to be resolved into two component velocities, if the lines representing the components form the two sides of a parallelogram, of which the diagonal represents the actual velocity.

15. (Art. 136.) The effect of a constant unbalanced force, which has acted for any time upon a body in motion, is measured by the energy exerted as regards the force, and by the change in the kinetic energy as regards the body.

16. (Art. 162.) **Conservation of Energy.**—In any system of matter in motion under its own forces, the total energy, potential and kinetic, of the system remains constant.

17. (Art. 174.) If any number of forces acting on one point are *in equilibrium*, then the algebraical sums of the components of the

forces, taken along any three rectangular axes, must separately vanish.

18. (Art. 189.) **Principle of the Lever.**—If two forces acting at the extremities of a lever, and tending to twist the lever opposite ways, are in equilibrium, the moments of the forces about the fulcrum are equal.

19. (Art. 193.) The proper measure of the effect of a force, in tending to produce rotation of a body about a fixed point, is the moment of the force about that point.

20. (Art. 203.) If any number of forces, acting on a body in one plane, are in equilibrium, then the sum of the components of the forces in any direction must be zero, and the sum of the moments of the forces round any point must be zero.

21. (Art. 204.) If any number of forces acting on a rigid body in one plane tend to turn it about a fixed axis, there will be equilibrium when the algebraical sum of all the moments about that axis is zero.

22. (Art. 209.) Every system of parallel forces has a centre; that is, a point such that, if it be fixed, the body will be in equilibrium in any position. When the forces are weights, this point is called the centre of gravity.

23. (Art. 229.) **First Law of Statical Friction.**—The limit of friction between two bodies at rest is proportional to the normal pressures between the bodies.

24. (Art. 230.) **Second Law of Statical Friction.**—The limit of friction is independent of the area of the surfaces in contact.

25. (Art. 234.) The reaction of a smooth surface is always normal to the surface.

26. (Art. 240.) If the external forces acting on a system are in equilibrium, and if the system receive a conservative displacement, the net energy exerted by the external forces will be zero.

27. (Art. 242.) **Principle of Virtual Velocities.**—If a system in equilibrium receive a very small conservative displacement, the sum of the virtual moments of all the external forces is zero.

28. (Art. 338.) If we suppose the velocity of any one particle in a system, reversed in direction, to be communicated to each particle of the system in addition to that which it already

possesses, then the relative motions of all about the first thus reduced to rest, will be the same as their relative motions about it when all were in motion.

29. (Art. 359.) A particle moving in a curved line without friction does not lose any of its velocity in consequence of the reactions which compel it to move in that line.

30. (Art. 367.) If a particle of mass m move in any curve whatever, it may be taken as acted on at any point by a centripetal force $= m\frac{v^2}{r}$, where v is the velocity, and r the radius of curvature at that point.

31. (Art. 393.) **First Law of Elasticity.**—The ultimate atoms of every body are perfectly elastic.

32. **Second Law of Elasticity.**—Every body, taken as a whole, is imperfectly elastic.

33. **Third Law of Elasticity.**—In any body, taken as a whole, the ratio which the force of restitution bears to the force of compression depends only on the nature of the body.

34. (Art. 437.) The actual energy of any rotating body, due to the rotation, is given by $\frac{I\omega^2}{2}$, where I is the moment of inertia round the axis, and ω the angular velocity.

PART VI.—APPENDIX OF EXAMPLES.

(*See Answers at end*, p. 208.)

I.—CONDITIONS OF EQUILIBRIUM.

There are two general methods of solving the large class of problems, which consist in determining the conditions of equilibrium of a given system in two dimensions. They are as follows.

Method 1. Resolve all the forces in two directions at right angles to each other; and equate their algebraical sum in each direction to zero. Then take the moments of all the forces about some point in the plane, and equate their algebraical sum to zero. There are thus three equations, algebraical or trigonometrical; and if there are not more than three independent unknown quantities in the system, the solution of these three equations will give all the conditions of equilibrium. If there are more than three such quantities, the conditions cannot be fully determined. If some of the unknown quantities are not independent, they can be reduced in number by forming the geometrical equations between them.

In working this method it is desirable to choose the directions of resolution, and also the point about which to take moments, so as to make the resulting equations as simple as possible. No general rules can, however, be laid down for this choice, which depends upon the circumstances of each problem. If there are several bodies in the system, we can give the equations for each, and use these to eliminate the unknown reactions existing between them.

Method 2. Assume the system to receive some small conservative displacement, *i.e.*, a displacement that will not alter the geometrical relations of the various forces concerned. Then find the energy exerted by each force during the displacement, and equate the algebraical sum of these to zero. This is generally called the forming of the equation of virtual velocities. If this equation can be so formed as to contain only one unknown quantity, that quantity can at once be determined.

Here again much will often depend on the choice of the particular displacement, which should be such that for all the unknown reactions, whose value there is generally no occasion to find, the virtual moment may vanish.

Of course the two methods may be combined, the equation of virtual velocities being formed, for instance, instead of the equation of moments. The equations, however, are not all independent of each other; so that it is

not possible thereby to solve a problem in which there are more than three unknown quantities.

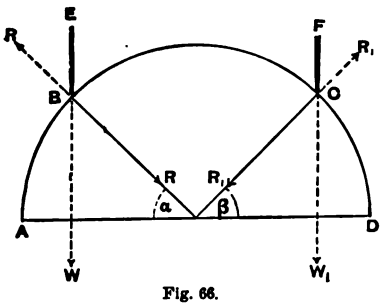

Fig. 66.

As a typical example of both methods, take the case of a smooth semicircle, A B C D, which can slide upon a horizontal plane A D, and is kept in equilibrium by two rods E B, F C, which slide vertically in smooth rings, and bear upon given points in the circumference.

First, by the method of Resolution. Let $A O B = \alpha$, $C O D = \beta$: let W, W_1, be the weights of the rods, and R, R_1, the unknown reactions at B and C, which, since the semicircle is smooth, must act along the normal lines O B, O C (Law No. 25). Consider the rod E B. It is kept in equilibrium by the weight W, the reaction R, and two unknown horizontal forces at the two rings. Resolving vertically, to avoid these, we have

$$W = R \sin \alpha.$$

By similar reasoning $W_1 = R_1 \sin \beta$.

Lastly, the semicircle is kept in equilibrium by the two pressures R, R_1, and by the vertical upward reaction of the plane. Resolving horizontally to avoid the latter, we have

$$R \cos \alpha = R_1 \cos \beta.$$

But

$$R \cos \alpha = \frac{W}{\sin \alpha} \cos \alpha$$

$$R_1 \cos \beta = \frac{W_1}{\sin \beta} \cos \beta:$$

Hence

$$W \cot \alpha = W_1 \cot \beta.$$

This gives the relation between W and W_1, so that if either is known the other can be found.

Secondly, by the method of virtual velocities. Assume the semicircle to be pushed along the plane, in direction O A, by a very small distance, d. Then, if H is the new position of B, it is easily seen that E B will be lifted through a space B H, where

$$B H = d \cot \alpha.$$

By similar reasoning, F C will be lowered by a distance $= d \cot \beta$.
Hence the equation of virtual velocities is

$$W d \cot \alpha = W_1 d \cot \beta.$$

Or,

$$W \cot \alpha = W_1 \cot \beta, \text{ as before.}$$

It is evident that the other forces involved will not appear in this equation: for the reactions R, R_1, are in equal and opposite pairs, the reactions of the rings are at right angles to the motion of the rods, and the upward reaction of the plane is at right angles to the motion of the semicircle.

[In this example the method of virtual velocities is much the shorter and simpler; and this will generally be the case. On the other hand, much more care is required in forming the equation. It is generally better to begin with an attempt to form an equation of virtual velocities, and if this appears to be difficult, then to turn to the more formal method of resolution.]

As another example, let us take the following.

Two equal rods, A B, A C, of weight W, are jointed together by a hinge at A, and united at their middle points by a weightless string, D E. They are placed on a horizontal plane, B C: find the forces acting at D or E.

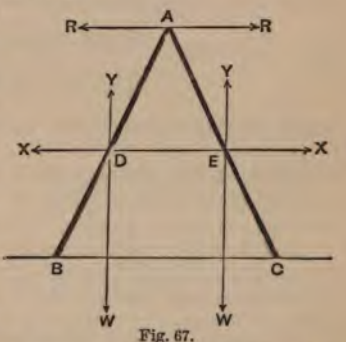

Fig. 67.

[In this case it is impossible to form the equation of virtual velocities, for we cannot make a displacement without either breaking or slackening the string. We must proceed by the method of Resolution.

We may first consider the direction of the reactions at the hinge A. A hinge is considered mathematically merely as an indefinitely short piece of string, uniting two rods together, so that they can turn freely about each other, but cannot be separated. The exact direction the string will take—which is the same thing as the direction of the reactions at the hinge—must be settled by the circumstances of each particular case. But there is one general rule on the subject, namely, that if there is any line about which the whole figure, and the forces acting upon it, are symmetrical, then the reactions must lie at right angles to that line. This is easily seen. For each reaction must be equal and opposite to the resultant of the forces on the opposite side of the line of symmetry, since it keeps these forces in equilibrium; and by symmetry these resultants must make equal angles with the line of symmetry: therefore the reactions do the same. And since they are equal and opposite to each other, they cannot make equal angles with any line, unless those angles be right angles: hence they are at right angles to the line of symmetry.]

Now, in the present case it is obvious that everything is symmetrical about the vertical line drawn through A. Hence the reactions R, R at A are horizontal. Let R_1, R_1 be the vertical reactions at B and C. By symmetry these will be equal to each other. Resolving vertically for the whole system, we have

$$2\,R_1 = 2\,W, \text{ whence } R_1 = W.$$

None of the reactions enter into this equation, because they are all in equal and opposite pairs.

Let X, Y be the horizontal and vertical components of the reaction at D, E. By symmetry they will be alike at each point. Now consider the equilibrium of the string D E. Resolving vertically

$$2\,Y = 0\,;\ \text{or}\ Y = 0.$$

Again, for the rod A B, resolving horizontally,

$$X - R = 0.$$

Take moments about the point B. Let $2a$ be the length of the rod, and let $BAC = 2\alpha$. Then the equation of moments is

$$X\,a\cos\alpha + R\,2a\cos\alpha = Wa\sin\alpha.$$

Or

$$X\,3a\cos\alpha = Wa\sin\alpha, \qquad X = \frac{W\tan\alpha}{3},$$

which gives the value of X.

Another class of problems are those where friction comes into the question. In these we are no longer concerned with the *absolute* position of equilibrium, but with the *limiting* position. This position will be such that the coefficient of friction in that position reaches the limiting value μ, which it cannot exceed without slipping taking place.

As an example, consider a circle, of radius r, which rests against a rough wall, and is supported by a string wrapped round it, and fastened to a nail in the wall at some distance above. Let θ be the angle which the string's direction makes with the vertical, R the horizontal reaction between the wall and the circle: then μ R is the friction acting vertically at the same point.

Resolving the forces horizontally, we have

$$T\sin\theta - R = 0.$$

Taking moments about the centre of the circle,

$$T\,r = \mu R\,r, \text{ or } T = \mu R.$$

Hence

$$\sin\theta = \frac{1}{\mu}$$

This gives the limiting value, which D must not exceed, or the circle will slip.

1. Two forces, of 3 lbs. and 4 lbs. respectively, act on a particle at right angles to each other: find the magnitude of their resultant.

2. Two forces, of 1 lb. and 2 lbs., act at an angle of 60°: find the direction and magnitude of their resultant.

3. Two forces, of 6 lbs. and 10 lbs., acting at a point, include an angle of 105°: find the resultant.

4. A man pulls a weight by means of a rope along a road, with a force of 20 lbs. The rope makes an angle of 30° with the road: find the force he would need to apply parallel to the road.

5. Three forces act at a point, and include angles of 90° and 45°. The first two forces are each equal to 2 P, and the resultant of them all is $\sqrt{10}$ P: find the third force.

6. A string, fixed at its extremities to two points in the same horizontal line, supports a ring weighing 10 oz.; the two parts of the string contain an angle of 60°: find the tension.

7. Show that three forces, acting at a point, but not in the same plane, cannot be in equilibrium.

8. The resultant of two forces that act at right angles to one another is 145 lbs., and one of the forces is 144 lbs.: find the other.

9. Three posts are placed in the ground so as to form an equilateral triangle, and an elastic ring is stretched round them, the tension of which is 6 lbs.: find the pressure on each post.

10. An even number of points are taken at equal distances on the circumference of a circle, and from one of the points straight lines are drawn to the rest: if these straight lines represent forces acting on the point, find the straight line which will represent their resultant, supposing their number to be given.

11. Two weights, W and 2 W, are connected by a rigid, weightless rod, and also by a loose string which is slung over a smooth peg: compare the lengths of the string on each side of the peg when the weights have assumed their position of equilibrium.

12. Two strings at right angles to each other support a weight, and one string makes an angle of 30° with the vertical line: compare the tensions of the strings.

13. A weight of 24 lbs. is suspended by two flexible strings, one of which is horizontal, and the other is inclined at an angle of 30° to the vertical direction: what is the tension in each string?

14. Suppose a person of given weight to be suspended in a scale from the extremity of an immovable beam, and to press upwards by means of a rod of a given length against the underside of the beam: with what force must he press upwards, so that he may rest in any given position?

15. A string of given length is suspended to two tacks anywhere situated, the length of the string being greater than the distance between them: it is required to find the position of equilibrium of a given weight (w), which slides freely on the string.

16. A uniform pole leans against a smooth wall, at an angle of 45°, the lower end being on a rough horizontal plane: show that the amount of friction to prevent sliding is half the weight of the pole.

17. If when two particles are placed on a rough double inclined plane, and connected by a string passing over a smooth peg at the vertex, they are on the point of motion; and if when their positions are interchanged no friction is called into play: show that the limiting angle of friction is equal to the difference of the inclinations of the two planes.

18. A heavy uniform beam rests with one end against a rough horizontal, and the other end against an equally rough vertical plane: find the least coefficient of friction that will allow the beam to rest in all positions.

19. A wheel, radius a, weight w, is acted upon by a weight W at its circumference, and turns round a rough fixed axle, radius b. Assuming the inside of the wheel and the axle only to touch in one point: find the greatest value of W under which the wheel will not turn.

II.—PARALLEL FORCES.

Ordinary cases of parallel forces are simply solved by the Equation of Moments. Thus, take the following example.

A man supports two weights, slung on the ends of a weightless rod, 40 inches long: if one weight be two thirds of the other, find the point of support.

Let W be one weight, a the distance from it to the point of support.

$\frac{2}{3}$W the other, $40-a$ " " "

Then, by the Equation of Moments

$$Wa = \tfrac{2}{3}W(40-a)$$

$$3a+2a = 80, \qquad a = 16.$$

Hence the support is 16 inches from the heavier weight.

Questions relating to the centre of gravity are rather exercises in Geometry than Mechanics. They all depend on the fundamental theory that the co-ordinates of the centre of gravity of any system of particles are given by the equations

$$\bar{X} = \frac{\Sigma m x}{\Sigma m}, \quad \bar{Y} = \frac{\Sigma m y}{\Sigma m}, \quad \bar{Z} = \frac{\Sigma m z}{\Sigma z}.$$

But the application of these principles, except in simple cases, requires the use of the Integral Calculus.

In many cases the principle of symmetry enables us to decide on the position of the centre of gravity—*e.g.*, it is evident that the centre of gravity of a sphere or a circle will be at its centre.

[There are several geometrical figures of which the centre of gravity has been determined, and should be known by the student. The principal ones are the following:—

Triangle. Draw a line from any angle to bisect the opposite side, the centre of gravity is two-thirds down the line from the angle.

Pyramid or Cone. Draw a line from the vertex to the centre of gravity of the base: the centre of gravity is on this line, at one-fourth of its length from the base.

Quadrilateral Figure. Draw a diagonal; find the centre of gravity of the two triangles thus formed, join these points, and divide the line joining them

so that the products formed by multiplying each segment by the area of the adjacent triangle are equal.

Circular Arc. Let 2α be the angle made at the centre, in circular measure. The centre of gravity is on the line joining the centre to the middle point of the arc, at a distance from the centre $= \frac{\sin\alpha}{\alpha} \times$ (radius).

Semicircle. The distance of the centre of gravity is given by $\frac{2\,(\text{radius})}{\pi}$

Circular Sector. The distance of the centre of gravity from the centre is given by $\frac{2}{3}\frac{\sin\alpha}{\alpha}$(radius).

Solid Semicircle. Distance is given by $\frac{4}{3}\frac{(\text{radius})}{\pi}$.

Hollow Hemisphere. Centre of gravity bisects the radius of symmetry.

Solid Hemisphere. Centre of gravity is on the radius of symmetry, at $\frac{3}{8}$ the distance from the centre.]

As an example, take the following. A quarter of a triangle is cut off by a straight line drawn parallel to one of the sides: to find the centre of gravity of the remaining piece.

From the common angle or vertex, draw a line to bisect the opposite side of the great triangle, or the base: then by similar triangles it will also bisect the opposite side of the small triangle. Hence the centre of gravity of both triangles lies on this line. And the centre of gravity of the remaining part of the great triangle will lie on it also, since it bisects all the elements into which that part may be divided by lines parallel to the base. Let x be the distance of this centre of gravity from the vertex, D the length of the whole line, d that of the part of it within the small triangle: then, taking moments about the centre of gravity of the great triangle,

$$\tfrac{3}{4}\,(x-\tfrac{2}{3}\,\mathrm{D}) = \tfrac{1}{4}\,(\tfrac{2}{3}\,\mathrm{D}-\tfrac{2}{3}\,d).$$

But in similar triangles, of which one is four times the other, it is easily seen that $d = \frac{1}{2}\,\mathrm{D}$: hence the equation becomes

$$3x - 2\,\mathrm{D} = (\tfrac{2}{3}\,\mathrm{D} - \tfrac{1}{3}\,\mathrm{D}) = \tfrac{1}{3}\,\mathrm{D}$$

$$x = \tfrac{7}{9}\,\mathrm{D}.$$

1. The ratio of two unlike parallel forces is $\frac{4}{5}$, and the distance between them is 10 inches: find the position of the resultant.

2. A plank, weighing 10 lbs., rests on a single prop at its middle point; if the prop be replaced by two others on each side of it, 3 feet and 5 feet from the middle point, find the pressure on each.

3. A horizontal straight bar, 6 feet long and weighing 16 lbs., is supported at each end, and a weight of 48 lbs. is hung at 2 feet from one end: find the pressure upon each of the supports.

4. Let a given weight (W) be placed at O on a triangle, ABC, and supported by three props, A, B, C, at the angles. The pressure upon each prop

is proportional to the area of the triangle opposite to it; that is, the pressure on A : pressure on B :: area of the triangle BOC : area of the triangle AOC.

5. A piece of uniform wire is bent into the shape of an isosceles triangle, of which each of the equal sides is 5 inches long, and the third side 6 inches: find its centre of gravity.

6. A triangle is suspended from the middle point of one side: find the weight that must be attached to one of the ends of the side, in order to make it rest with that side horizontal.

7. The middle points of two adjacent sides A B, A D of a square are joined, and the triangle formed by this line and the edges is cut off: find the centre of gravity of the remainder of the square.

8. A wire is bent into the form of a triangle, and hangs from one angle with the base horizontal: show that the triangle is isosceles.

9. A sugar-loaf stands on an inclined plane, rough enough to prevent sliding, whose inclination to the horizon is 45°: show that it will fall over if the height of the sugar-loaf be more than twice as great as the diameter of its base; the centre of gravity of a cone being distant from the base one-fourth of the height.

10. A heavy pole, weighing 12 lbs., whose centre of gravity falls at a distance of one-fourth of the pole from one end, is fastened at this end by a rope to the ceiling, and the other end is raised by a man, so that the pole is horizontal: what weight will the man support?

11. A circular tower, the diameter of which is 20 feet, is being built, and for every foot it rises it inclines 1 inch from the vertical: what is the greatest height it can reach without falling?

12. An equilateral triangle stands vertically on a rough plane: find the ratio of the height to the base of the plane when the triangle is on the point of overturning.

13. From a circular disc, another disc, having for its diameter the radius of the first circle, is cut away: find the centre of gravity of the remainder.

14. Find the height of a cylinder which can just rest on an inclined plane, the angle of which is 60°, the diameter of the cylinder being 6 inches.

15. A cylindrical vessel, weighing 4 lbs., and the internal depth of which is 6 inches, will just hold 2 lbs. of water. If the centre of gravity of the vessel when empty is 3·39 inches from the top, determine the position of the centre of gravity of the vessel and its contents when full of water.

III.—MECHANICAL POWERS.

Problems on Mechanical Powers are merely exercises in the application of the equations between P and W deduced in the text for each kind of power: the conditions of equilibrium having thus been determined already for each separate case.

Problems regarding false balances depend on the following theorem, which *we may take* as an example.

If a substance be weighed in a balance having unequal arms, and apparently weighs p lbs. in one, and q lbs. in the other, its true weight is $\sqrt{pq}$ lbs.

Let L l be the two arms, w the true weight: then, taking moments about the centre of suspension we have in the two cases

$$w\,\mathrm{L} = p\,l$$
$$w\,l = q\,\mathrm{L};$$

multiplying

$$w^2 = p\,q, \text{ or } w = \sqrt{pq}.$$

The following is an example of another class. The difference of the diameters of a wheel and axle is 2 feet 6 inches, and the weight = 6 times the power: find the radii.

Let r be the radius of the axle in inches.

Then $r+15$ is ,, wheel ,,

Hence the equation of moments becomes

$$\mathrm{W}\,r = \mathrm{P}\,(r+15),$$

and W = 6 P: hence $6r = r+15$, $r = 3$.

Hence the radii are 3 inches and 18 inches respectively.

1. A man exerts a pressure of 50 lbs. on a crowbar at a distance of 4 feet from the fulcrum: what weight will he balance at 3 inches from the fulcrum?

2. In a lever of the second kind, W = 11 lbs., W's arm = 16 inches, P's arm = 100 inches: find the value of P.

3. In a lever of the third kind, W = 40 lbs., W's arm = 60 inches, and P's arm = 8 inches: find the value of P.

4. A lever with a fulcrum at one end is 3 feet in length; a weight of 28 lbs. is suspended from the other end: if the weight of the lever is 2 lbs. at its middle point, at what distance from the fulcrum will an upward force of 50 lbs. preserve the equilibrium?

5. Suppose the arms a, b of a bent lever, inclined to each other at a given angle α at the fulcrum, to be also given: required the position of the lever when two given weights P, Q, suspended freely from its extremities, balance each other.

6. Two weights, of 6 lbs. and 8 lbs., are hung from the ends of a lever 7 feet long: where must the fulcrum be placed so that they may be balanced?

7. A beam, the length of which is 12 feet, balances at a point 2 feet from one end; but if a weight of 100 lbs. be hung from the other end, it balances at a point 2 feet from that end: find the weight of the beam.

8. If the pressure on the fulcrum be 5 lbs., and one of the weights be distant from the fulcrum one-sixth of the whole length of the lever: find the weights.

9. Two weights, 12 lbs. and 8 lbs. respectively, at the ends of a horizontal lever, 10 feet long, balance: how far ought the fulcrum to be moved for the weights to balance when each is increased by 2 lbs?

10. A heavy pole, weighing 12 lbs., whose centre of gravity falls at a distance of one-third of the pole from one end, is carried horizontally by two men, one at each end: find the weight supported by each man.

11. A rod, AB, moves about a fixed point, B. Its weight, W, acts at its middle point, and it is kept horizontal by a string, AC, that makes an angle of 45° with it: find the tension in the string.

12. In a steelyard, the weight of the beam is 10 lbs., and the distance of its centre of gravity from the fulcrum is 2 inches: find where a weight of 4 lbs. must be placed to balance it.

13. A weight weighs 16 lbs. when put in one scale of a false balance, and only 9 lbs. when in the other scale: find the weight.

14. Given the two apparent weights in each end of a pair of false scales: show whether the true weight is greater or less than half their sum.

15. Is the mechanical advantage of a wheel and axle increased or diminished by lessening the radii of wheel and axle by the same amount?

16. If the radii of a wheel and axle be as 10 to 4, and weights of 3 oz. and 8 oz. hang from them: which will descend?

17. If the radius of the wheel is three times that of the axle, and the string round the wheel can support a weight of 40 lbs. only: find the greatest weight that can be lifted.

18. A wheel and axle is used to raise a bucket from a well. The radius of the wheel is 15 inches, and while it makes seven revolutions, the bucket, which weighs 30 lbs., rises 5½ feet: show what is the smallest force that can be employed to turn the wheel.

19. A man, whose weight is 12 stone, has to balance by his weight 15 cwt.: show how to construct a wheel and axle which would enable him to do this.

20. What must be the radius of the wheel to enable a power of 1½ lbs. to raise a weight of 2 stone, the diameter of the axle being 6 inches?

21. What force is necessary to raise a weight of 120 lbs. by an arrangement of six pulleys, in which the same string passes round each pulley?

22. If there be equilibrium between P and W, with three pulleys, in that system in which each string is attached to the weight, what additional weight can be raised if 2 lbs. be added to P?

23. A man, weighing 150 lbs., raises a weight of 4 cwt. by a system of four movable pulleys arranged according to the first system: what is his pressure on the ground?

24. In a system of pulleys, in which the same string passes round all the pulleys, a man, whose weight is 12 stone, can support 18 cwt.: how many pulleys must there be on the lower block?

25. In a system of one fixed and four movable pulleys, in which one end of each string is fixed to a beam: find the relation between the power and the weight (neglecting the weights of the pulleys), when one of the strings is nailed to the pulley round which it passes.

26. The angle of a plane is 45°: what weight can be supported by a horizontal force of 3 oz. and a force of 4 oz. parallel to the plane, both acting together?

27. What is the horizontal force necessary to support a weight of 1 lb. on a plane that rises 3 in 5?

28. *What* weight can be supported on a plane by a horizontal force of 10 oz., *if the* ratio of height to base is ¾?

29. If a weight, W, be supported on an inclined plane by a force, $\frac{W}{2}$, parallel to the plane: what is the inclination of the plane?

30. A weight (P) draws another weight (W) along an inclined plane of angle α, given in position, by means of a rope passing over a fixed pulley: it is required to find the position of the weight when in equilibrium.

31. A globe of given weight is supported between two planes inclined to the horizon at the respective angles of 60° and 30°: compare the weights sustained by the planes with each other and with the whole weight.

32. A sphere rests between two inclined planes, and the pressure on one, which is given in position, is half that on the other plane: construct the position of the other plane.

33. In a screw, which has seven threads to the inch, find the pressure (neglecting friction) that can be produced by a force of 6 lbs. applied at the circumference, the radius of the cylinder being 1 inch.

34. How many turns must be given to a screw formed upon a cylinder, whose length is 10 inches and circumference 5 inches, that a power of 2 oz. (neglecting friction) may overcome a pressure of 100 oz.?

35. If a power of 1 lb. describe a revolution of 3 feet, whilst the screw moves through $\frac{1}{4}$ inch: what pressure (neglecting friction) will be produced?

IV.—ACCELERATING FORCES.

Examples of this class are merely exercises in the application of the three fundamental equations given in Art. 90 (Law No. 10).

The following is an example.—A body moving from rest is observed to move over 80 and 112 ft. during two consecutive seconds: find the acceleration and the time from rest.

Let t be the time up to the first of these seconds, f the acceleration: then the space described during that second is

$$ft+\frac{f}{2}=80.$$

The space described during the next second is

$$f(t+1)+\frac{f}{2}=112.$$

Hence $f=32$, or the difference between the spaces described in two consecutive seconds is equal to the acceleration.

Also, from the first equation,

$$t=\frac{80-16}{32}=2 \text{ seconds.}$$

Again, to find a point in a vertical circle, such that the time down a tangent at that point terminating in the vertical diameter produced, may be equal to the time down the vertical diameter.

Let θ be the angle the tangent makes with the vertical.

Then $r \cot \theta$ is the length of the tangent, and $g \cos \theta$ is the acceleration. Hence the time down the tangent is given by

$$r \cot \theta = \frac{g \cos \theta}{2} t^2.$$

But the time down the vertical diameter is given by

$$2r = \frac{g t^2}{2},$$

dividing

$$\frac{\cot \theta}{2} = \cos \theta, \text{ or } \sin \theta = \tfrac{1}{4}: \quad \theta = 30°.$$

1. A body is projected upwards with a velocity of $5g$ feet: to what height will it rise?

2. A body projected vertically upwards passes a certain point with a velocity of 80 feet per second: how much higher will it ascend?

3. A body starts with a velocity of 90 feet, and loses a third of its velocity per second: how far will it move?

4. Two bodies are let fall at an interval of one second: find how far they will be apart after a lapse of four seconds from the fall of the first.

5. The space described by a heavy body in the fourth second, is to the space described in the last second, except 4, as 1 to 3: required the whole space described.

6. With what velocity must a body start, if its velocity be retarded 10 feet per second, and it come to rest in 12 seconds?

7. A body moving from rest with a uniform accleration, describes 90 feet in the fifth second of its motion: find the acceleration, and the velocity after 10 seconds.

8. A body, whilst moving vertically downwards with a uniform velocity of 10 feet per second, is urged horizontally with an acceleration of 5 feet per second: find its distance from starting point after 2 seconds.

9. A body, moving with a uniform velocity of 30 miles an hour, has its velocity accelerated 10 feet per second in the same direction: find the space traversed in a quarter of a minute.

10. A body has fallen from a feet when another body is let fall from a point b feet below it: how far will the latter body fall before it is overtaken by the former?

11. Let two bodies, with the velocities v, V, be projected at the same time, and towards each other, from the two extremities of a vertical line; then they shall always meet in the middle of the line, if the difference of the squares of the velocities v, V, is equal to the square of the velocity acquired in falling down half the line.

12. If two bodies are moved at the same time towards each other, from the two extremities of a vertical line l, one projected upwards with a velocity acquired down $\frac{3}{2}l$, the other let fall from rest: it is required to find the *point* where they meet.

13. A body is projected up an inclined plane, whose length is ten times its

height, with a velocity of 30 feet in one second: in what time will its velocity be destroyed?

14. A body falls from rest by the force of gravity down a given inclined plane: compare the times of describing the first and last halves of it.

15. Two bodies, projected down two planes inclined to the horizon at angles of 45° and 60°, describe in the same time spaces respectively as $\sqrt{2}$: $\sqrt{3}$: required the ratio of the initial velocities of the projected bodies.

16. Through what chord of a circle must a body fall to acquire half the velocity gained by falling through the diameter.

17. Of right-angled triangles having the same base, determine that down whose hypothenuse a body descends in the least time.

18. Find the time of describing 30 feet on a plane inclined to the horizon at an angle of 30°, the force of gravity being supposed to be diminished by one-fourth of its present quantity.

19. Determine that part of an inclined plane A C, through which a body will descend from rest in the time of falling from the top A to the bottom at B.

V.—PROJECTILES.

Examples in projectiles are best solved, in almost all cases, by at once resolving the velocity of projection into two parts, vertical and horizontal. Of these the first is acted on exclusively by gravity, and comes under the general laws of accelerating forces, as given in Law No. 10: while the latter remains constant throughout the motion.

Take the following. A ball fired with velocity u, at an angle α to the horizon, just clears a vertical wall, which subtends an angle β at the point of projection: determine the time at which the ball passes the wall.

Let t be the time, and resolve the velocity into $u \cos \alpha$ and $u \sin \alpha$ respectively. Let x be the distance to the foot of the wall, and y its height. Then we must have

$$(u \cos \alpha)\, t = x.$$

Also the vertical height described in time t is

$$(u \sin \alpha)\, t - \frac{g}{2} t^2 = y,$$

$$\text{and } y = x \tan \beta.$$

Substituting from the last equation in the second, and dividing by the first, we have

$$\frac{u \sin \alpha - \frac{g t}{2}}{u \cos \alpha} = \tan \beta\,; \text{ or } t = \frac{2u}{g} \frac{\sin (\alpha - \beta)}{\cos \beta}.$$

Again, in the same case, to determine the distance between the foot of the wall and the point where the ball strikes the ground. The time of flight is given by the equation,

$$(u \sin \alpha)\,T - \frac{g}{2}T^2 = 0, \text{ or } T = \frac{2u \sin \alpha}{g},$$

and the range, or horizontal distance described, is therefore

$$\frac{2u \sin \alpha}{g} u \cos \alpha = \frac{u^2 \sin 2\alpha}{g}.$$

But the distance to the foot of the wall $= (u \cos \alpha)\, t$

$$= (u \cos \alpha)\frac{2u}{g}\frac{\sin(\alpha - \beta)}{\cos \beta}.$$

The difference required is therefore

$$\frac{2u^2 \cos \alpha}{g}\left(\sin \alpha - \frac{\sin \alpha - \beta}{\cos \beta}\right) = \frac{2u^2 \cos \alpha}{g}\frac{\cos \alpha \sin \beta}{\cos \beta}.$$

N.B.—In these examples, g may be taken as 32.

1. A body is projected horizontally with a velocity of 4 feet per second: find the *latus rectum* of the parabola described.

2. Find the direction of projection which gives the greatest range on a horizontal plane.

3. Find any two angles of elevation which give the same range on a horizontal plane through the point of projection.

4. A body projected from the top of a tower at an angle of 45° above the horizontal direction, fell in 5 seconds at a distance from the bottom of the tower equal to its altitude: find the altitude in feet.

5. A body with a velocity of 80 feet per second is projected upwards, along a plane of 50 feet long, and inclined to the horizon at an angle of 30°; after quitting the plane, it describes a parabola: find the parameter.

6. Let a body be projected from the top of a tower horizontally, with the velocity gained in falling down a space equal to the height of the tower: at what distance from the base of the tower will it strike the horizon?

7. Find the time of flight of a body projected at an angle of 30° to the vertical, with a velocity of 193 feet in a second, before striking a plane inclined at an angle of 60°.

8. Find the velocity and time of flight of a body projected from one extremity of the base of an equilateral triangle, and in the direction of the side adjacent to that extremity, to meet an object placed in the other extremity of the base.

9. Given v the velocity of sound: find the horizontal range, when a ball, at a given angle α, is so projected towards a person that the ball and sound of the discharge reach him at the same instant.

VI.—MOVING FORCES.

Examples in moving forces are best solved by remembering that if P be *the moving* force and M the mass moved, then $\frac{P}{M}$ is the acceleration. If

W, the weight of the body moved, be given, then $\frac{W}{g}=M$, and hence $\frac{Pg}{W}$ is the acceleration.

Thus, suppose a weight of W lbs. to be placed on a plane, which is made to ascend vertically with an acceleration f: to find the pressure on the plane.

Let P be the pressure required, then since this is opposed by the weight of the body, P − W must be the moving force or unbalanced effort, hence $\frac{(P-W)g}{W}$ is the acceleration. But this acceleration is f: hence,

$$f=\frac{(P-W)g}{W},$$

$$P=\frac{(f+g)W}{g}.$$

Again, take the following:—A force acting uniformly during $\frac{1}{10}$ second, produces in a given body a velocity of 1 mile per minute; compare the force with the weight.

Let W be the weight of the body, P the force. Then $\frac{P}{W}g$ is the acceleration. Also the velocity acquired, in feet per second $=\frac{3\times1760}{60}$. Hence, the formula $v=ft$ becomes

$$\frac{3\times1760}{60}=\frac{P}{W}\times32{\cdot}2\times\frac{1}{10}$$

$$\frac{P}{W}=\frac{880}{32{\cdot}2}=27{\cdot}3.$$

In cases of constrained motion, the value obtained for the centrifugal force—viz., $\frac{mv^2}{r}$—must be brought in. Take the following:—

A stone of 1 lb. is whirled horizontally by a string 2 yards long, having a tension of 3 lbs.: find the time of revolution.

Here, $m=\frac{1}{g}$, $r=6$ feet, centrifugal force $=3$ lbs.; hence, we have

$$\frac{1}{g}\ \frac{v^2}{6}=3 \qquad v^2=18g.$$

$$\text{Time of revolution}=\frac{2\pi r}{v}=2\pi\frac{6}{\sqrt{18g}}=2\pi\sqrt{\frac{2}{g}}.$$

N.B.—We may take $g=32$.

1. For how long a time must a force of 2 oz. act on a mass of 4 lbs., that it may give to the mass a velocity of 60 feet per second?

2. Through what distance must a force of 3 oz. act on a mass of 16 oz., to give to it a velocity of 6 feet per second?

3. A locomotive is moving at the rate of 20 miles an hour when the steam

is let off; if the force of friction be equivalent to $\frac{1}{100}$ of the weight of the engine, after what time will it stop?

4. A weight of 1 cwt. goes up and down on a lift with a uniform acceleration of 4 feet per second: find the pressure of the weight on the lift in each case.

5. Show how the third law of motion holds in the case of a stone which is in the act of falling towards the earth.

6. A and B hang over a pulley, and $A = 2B$: through what space will a body descend by the action of gravity whilst A descends 2 feet?

7. Two bodies, P and Q, each 1 lb. in weight, balance on a single pulley; an ounce weight is added to P: how long is it in descending through 12 feet, and what velocity does it acquire?

8. Two equal weights, P and Q, are connected by a string passing over a fixed pulley: what weight added to P will cause it to acquire in 6 seconds a velocity of 48 feet, and through what space will it have fallen in that time?

9. Let a weight, P, fastened to a string going over a wheel, by its descent cause two weights, q, q', to be wound upon two axles: required the velocity of P, after that it has descended through a space, s; the radii of the wheel and of the two axles being r, r', r''.

10. A body, whose weight is 10 lbs., is whirled round in a circle of 10 feet radius with a velocity of 30 feet per second: find the tension of the string.

11. The diameter of a circle is 10 feet: find the time of a revolution when the centripetal force = the weight.

12. If a body move in a vertical circle, the radius of which is 5 feet: determine the velocity at the highest point that the body may just keep in the circle.

13. A body revolves in a circle of radius, r, with centripetal force, P: if U is the potential energy of the body, show that $2U = Pr$.

14. A stone is whirled round horizontally by a string 2 yards long: what is the time of one revolution, when the tension of the string is four times the weight of the stone?

15. Let the earth be supposed a sphere of given magnitude, and to revolve about its axis in a given time: compare the weight of a body at its equator with its weight in a given latitude.

16. Compare the space described in one second by the force of gravity in any given latitude with that which would be described in the same time, if the earth did not revolve round its axis.

17. The quadrant A C is a thin rod, which revolves on the axis A B, perpendicular to the horizon; a ring moves freely on the rod: with what velocity must the point C revolve in order that the ring may always remain in the middle of the arc?

VII.—IMPULSIVE FORCES.

In examples on direct impact, it is generally simpler to determine first the *value of* R, from the equation $R = \frac{mM(U-u)}{M+m}$, and then treat each body

separately as having had impressed upon it a velocity $=\frac{(1+e)R}{\text{mass}}$. Its subsequent motion is then easily determined. Thus, take the following:—

A body impinges with velocity V on a body of half its weight having velocity v: find the velocities after impact if $e=\frac{1}{2}$.

Let 2 M and M be the bodies, then the value of R is

$$\frac{2M\times M}{2M+M}(V-v)=\frac{2M}{3}(V-v).$$

Hence 2 M's velocity after impact is

$$V-\frac{(1+\frac{1}{2})\frac{2M}{3}(V-v)}{2M}=V-\frac{V-v}{2}=\frac{V+v}{2};$$

and M's velocity after impact is

$$v+\frac{(1+\frac{1}{2})\frac{2M}{3}(V-v)}{M}=v+V-v=V.$$

Again, in cases where the change of momentum for one of the bodies is given, we know that that of the other must be equal and opposite to it; hence if the mass be known its change of velocity is easily found, or *vice versa*. Thus, suppose that a wagon A, weighing a tons, moving at c miles an hour, strikes another wagon B, weighing b tons, and is brought to rest; to find the velocity with which B will start.

The momentum of A is $\frac{ca}{g}$, and this is entirely destroyed; hence if v be B's velocity, we have

$$\frac{bv}{g}=\frac{ac}{g}, \text{ whence } v=\frac{ac}{b}.$$

1. If a weight of 2 lbs., at a velocity of 20 feet per second, overtakes one of 5 lbs., moving with a velocity of 5 feet per second: determine the common velocity after impact.

2. If the same bodies *met* with the same velocities: determine the common velocity after impact.

3. An arrow shot from a bow starts off with a velocity of 120 feet per second: with what velocity will an arrow twice as heavy leave the bow, if sent off with three times the force?

4. Two balls, weighing 8 oz. and 6 oz. respectively, are simultaneously projected upwards, and the former rises to a height of 324 feet and the latter to 256 feet: compare the forces of projection.

5. A perfectly elastic ball, M_1, impinges upon another, M_2, and this on a third, M_3: compare the velocity communicated to the third, with that which would have been communicated if the first had impinged upon it.

6. Two bodies, perfectly inelastic, of different masses, are moving towards each other, with velocities of 10 feet per second and 12 feet per second respec-

tively, and continue to move after impact with a velocity of 1·2 feet per second in the direction of the greater: compare their masses.

7. If a shot, weighing 20 lbs., leave a gun weighing 3 tons, with a velocity of 1200 feet per second: find the velocity of the gun's recoil.

8. A rifle is pointed horizontally, with its barrel 5 feet above a lake: when discharged the ball is found to strike the water 400 feet off: find approximately the velocity of the ball.

9. In elastic bodies, show that if A overtake B, the velocity of B after impact is greater than the velocity of A.

10. If two imperfectly elastic balls strike each other: prove that the relative velocity before impact is to the relative velocity after, as perfect to imperfect elasticity.

11. A ball, whose elasticity : perfect elasticity :: $n : 1$, falls from a given height h upon a hard plane, and rebounds continually till its whole motion is lost: find the space passed over.

VIII.—ENERGY AND WORK.

Questions involving energy and work are very often of great practical importance, but they are usually solved very simply, by care in applying the definitions and principles laid down in the book. Take the following.

How many horse-power are required to raise 120 tons of coal per hour from a pit 660 feet deep?

Here the foot-pounds of work required per hour $= 120 \times 2240 \times 660$.

Hence ,, ,, minute $= 2 \times 2240 \times 660$.

And 33,000 foot-lbs. per minute is one horse-power.

Hence the horse-power required $= \dfrac{2 \times 2240 \times 660}{33{,}000} = 99{\cdot}6$.

In cases of accumulated energy, the energy must be found by the expression $\dfrac{w v_2}{2g}$, where w must be in pounds, and v in feet per minute. The result will be in foot-pounds, and this must be equal to the work to be done.

Thus, take a train weighing 160 tons, and having a velocity of 48 feet per second: find how far it will run of itself before stopping, on a level road, frictional resistance 9 lbs. per ton.

The expression $\dfrac{w v^2}{2g}$ here becomes

$$\frac{2240 \times 160 \times 48 \times 48}{2 \times 32} = 2240 \times 160 \times 6 \times 6.$$

If n be the number of feet run, then $9 \times 160 \times n$ is the number of foot-pounds expended against friction when the train comes to rest; hence

$$9 \times 160 \times n = 2240 \times 160 \times 6 \times 6$$

$$n = 2240 \times 4 = 8960 \text{ feet.}$$

1. What are the units of work expended in raising a weight of 50 lbs. to a height of 31 feet?

2. The ram of a pile-driving engine weighs half-a-ton, and has a fall of 17 feet: how many units of work are performed in raising it?

3. If the weight of a man be 183 lbs., and he ascends a perpendicular height of 20 feet, what are the number of units of work done?

4. In what time will an engine of 10 H.-P. raise 5 tons from the depth of 132 feet?

5. From what depth will an engine of 28 H.-P. raise 11 tons in a minute?

6. A man can do 900,000 units of work in a day of 9 hours: at what fraction of a H.-P. does he work on an average?

7. It is said that a horse, walking at the rate of $2\frac{1}{2}$ miles per hour, can do 1,650,000 units of work in an hour: what force in pounds does he continually exert?

8. An engine is required to raise a weight of 13 cwt. from a depth of 280 yards in three minutes: what is its horse-power?

9. A locomotive draws a gross load of 60 tons, at 20 miles an hour; the resistances are at the rate of 8 lbs. per ton: what must be the H.-P. of the engine?

10. What is the gross weight of a train, which an engine of 25 H.-P. will draw at the rate of 25 miles an hour, the resistances being 8 lbs. per ton?

11. How much work is done per hour, if 100 lbs. be raised 3 feet in one minute?

12. A body weighing 40 lbs. is projected along a rough horizontal plane, with a velocity of 150 feet per second; the co-efficient of friction is one-eighth: find the work done against friction in five seconds.

13. A body weighing 50 lbs. is projected along a rough horizontal plane, with a velocity of 40 yards per second: what amount of work is expended when the body comes to rest?

14. When a particle which is acted upon by any number of forces, moves during their action with uniform velocity in a straight line: show what condition the forces must fulfil.

15. The weight of a fly-wheel is 8000 lbs., the diameter 20 feet, diameter of axle 14 inches, co-efficient of friction 0·2; if the wheel is separated from the engine when making 27 revolutions per minute, find how many revolutions it will make before it stops (g taken $= 32{\cdot}2$.)

16. An engine of 35 H.-P. makes 20 revolutions per minute, the weight of the fly-wheel is 20 tons, and diameter 20 feet: what is the accumulated energy in foot-pounds?

17. If the fly-wheel in the last example lifted a weight of 4000 lbs. through 3 feet, what part of its angular velocity would it lose?

18. If the axis of the above fly-wheel be 6 inches diameter, co-efficient of friction 0·075, what is approximately the fraction of the 35 H.-P. expended in turning the fly-wheel?

ANSWERS TO EXAMPLES.

SECTION I.

(1.) 5 lbs. (2.) $\sqrt{7}$ lbs., acting at angle $\sin^{-1}\frac{\sqrt{3}}{7}$ (3.) 10·2 lbs. (4.) 17·32 lbs. (5.) P $\sqrt{2}$. (6.) $\frac{10}{\sqrt{3}}$ oz. (8.) 17 lbs. (9.) 6 $\sqrt{3}$. (10.) Resultant acts along diameter, and $= n \times$ diameter. (11.) One length double the other. (12.) $\sqrt{3}$ to 1. (13.) 8 $\sqrt{3}$ and 16 $\sqrt{3}$. (14.) If α, β be the angles made by the string and lever with the horizon, $R = W\frac{\cos\alpha}{\sin(\alpha-\beta)}$. (15.) The two parts of the string are equally inclined to the vertical. (18.) Unity. (19.) $w\left(\frac{a}{b}\sin\theta - 1\right)$, where θ is limiting angle of friction.

SECTION II.

(1.) 40 inches from the greater. (2.) $6\frac{1}{2}$ and $3\frac{1}{2}$. (3.) 40 and 24 lbs. (5.) On the line from the vertex bisecting the base, at $2\frac{1}{4}$ inches from the vertex. (6.) Let B be the point from which P hangs, A B the horizontal side, O the point of bisection, G the centre of gravity: then $P = W\frac{GO}{OB}\cos AOC$. (7.) If O is the intersection of diagonals, G the centre of gravity, then $OG = \frac{2}{21}OA$. (10.) 3 lbs. (11.) 240 feet. (12.) $\sqrt{3}$ to 1. (13.) At $\frac{1}{6}$ the radius from the centre. (14.) 3·464 inches. (15.) 3·26 inches from top.

SECTION III.

(1.) 800 lbs. (2.) 1·76 lbs. (3.) 300 lbs. (4.) 1·74 feet from fulcrum. (5.) $\text{Tan}\,\theta = \frac{Q\,b\cos\alpha - P\,a}{Q\,b\sin\alpha}$, where θ is angle which a makes with horizon. (6.) 3 feet from 8 lbs. weight. (7.) 25 lbs. (8.) $\frac{25}{6}$ and $\frac{5}{6}$. (9.) 2 inches. (10.) 8 lbs. and 4 lbs. (11.) $\frac{W}{\sqrt{2}}$. (12.) At 5 inches. (13.) 12 lbs. (14.) Less. (15.) Increased. (16.) The 8 oz. (17.) 120 lbs. (18.) 3 lbs. (19.) Radii as 10 to 1.

(20.) 56 inches. (21.) 20 lbs. (22.) 14 lbs. (23.) 122 lbs. (24.) 6. (25.) 1 to 8. (26.) 12 oz. (27.) $3+4\sqrt{2}$. (28.) $13\frac{1}{2}$ oz. (29.) 30°. (30.) Cos $\theta = \frac{W \sin \alpha}{P}$, where θ is angle between rope and plane. (31.) $R : R' : W :: 1 : \sqrt{3} : 2$. (32.) Sin $\theta = \frac{1}{2} \sin \alpha$. (33.) 264 lbs. (34.) 100. (35.) 144 lbs.

SECTION IV.

(1.) $\frac{25}{2} g$. (2.) 100 feet. (3.) 135 feet. (4.) $\frac{79}{2}$ (5.) 3600, if $g = 32$. (6.) 120 feet per second. (7.) 20 feet and 200 feet. (8.) $10\sqrt{5}$ feet. (9.) 1785 feet. (10.) $\frac{b}{\sqrt{2ga}}$. (12.) $\frac{l}{6}$. (13.) $9\frac{3}{8}$ seconds, if $g = 32$. (14.) As $1 : \sqrt{2}+1$. (15.) $\sqrt{2} : \sqrt{3}$. (16.) That of which the cord is inclined at 60° to the vertical. (17.) The isosceles triangle. (18.) $t = \sqrt{5}$. (19.) From A to the foot of the perpendicular let fall on the plane from B.

SECTION V.

(1.) 1 foot. (2.) 45°. (3.) α and $90° - \alpha$. (4.) $h = 200$ feet. (5.) $112\frac{1}{2}$. (6.) $x = 2h$. (7.) $\frac{193}{16\sqrt{3}}$. (8.) $u = \sqrt{\frac{2ag}{\sqrt{3}}}$, $T = \sqrt{\frac{3a}{g}}$. (9.) $\frac{2V^2}{g} \tan \alpha$.

SECTION VI.

(1.) 1 minute. (2.) 3 feet. (3.) $\frac{880}{3}$ seconds. (4.) $113\frac{1}{7}$ lbs. : $110\frac{6}{7}$ lbs. (6.) 6 feet. (7.) $t = \sqrt{\frac{99}{4}}$, $v = \frac{16}{\sqrt{11}}$. (8.) 144 feet.

$$(9.)\ v^2 = \frac{Pr - qr' - q'r''}{Pr + qr'^2 + q'r''^2} 2rgs.$$

(10.) 28 lbs. (11.) 2·445″. (12.) 12·65 feet per second. (14.) $\pi\sqrt{\frac{6}{g}}$.

(15.) $\frac{M - \frac{4\pi^2 R^3}{\tau^2} \cos^2 \theta}{M - \frac{4\pi^2 R^3}{\tau^2}}$, where M=mass of earth, R=radius, τ=periodic time.

(16.) $\frac{M - \frac{4\pi^2 R^3}{\tau^2} \cos^2 \theta}{M}$. (17.) $\sqrt{gr\sqrt{2}}$.

Section VII.

(1.) $9\frac{2}{7}$ feet per second. (2.) $2\frac{1}{7}$ feet per second in A's direction. (3.) 180 feet per second. (4.) As 3 to 2. (5.) $\frac{2M_2(M_1+M_2)}{(M_2+M_3)(M_1+M_3)}$. (6.) As 3 to 2. (7.) $3\frac{4}{7}$ feet per second. (8.) 715 feet per second. (11.) $h\frac{1+n}{1-n}$.

Section VIII.

(1.) 1550. (2.) 19040. (3.) 3660. (4.) 4·43 minutes. (5.) $37\frac{1}{2}$ feet. (6.) $\frac{5}{99}$. (7.) 125 lbs. (8.) 12·35 H.P. (9.) 25·6 H.P. (10.) 46·875 tons. (11.) 18,000 foot-lbs. (12.) 3500 foot-lbs. (13.) 11,250 foot-lbs. (14.) Equilibrium. (15.) 16·9 revolutions. (16.) 307,000. (17.) $\frac{1}{51}$. (18.) $\frac{1}{11}$.

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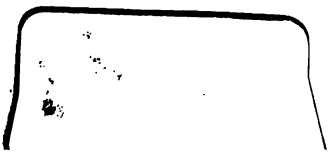
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